Pinter Consulting
New Series No. 20.
Spartan Old School Tutorials

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January 15, 2020

Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

- **"Der springt noch auf!"**
- Private studies for professional development:
- Socratic Programme
  - Analysis
  - Algebra and Number Theory
  - Geometry
  - Differential and Integral Equations
- Continuos improvement, corrections and last revision January 15, 2020.
- " A ló meghal a madarak kirepülnek
  bizonyos hogy a költő vagy épít magának valamit amiben kedve telik
  vagy bátran elmehet szivarvégszedőnek
  vagy
  vagy
  madarak lenyelték a hangot
  a fák azonban tovább énekelnek
  ez már az öregség jele
  de nem jelent semmit
  én Dr Melvin No vagyok
  s fejünk fölött elrölpül a nikkel szamovár. "
  (KASSÁK, 1922, edited)

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20.0 Session 7.

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- Ordinary Differential Equations
- *Spartan Old School*
- Last revision January 15, 2020

Course on Euler Math Toolbox

The first half of the fourth quarter of 2019 was dedicated to an introductory course on EMT. (Euler Math Toolbox x64 for Windows, Copyright Freiherr Rene Grothmann, GPL- Open Source, Version 2019-06-27. www.euler-math-toolbox.de)

EMT is an integrated math software package that combines numeric, symbolic as well as statistical and graphics applications. It contains Maxima, Python and Latex, and it is similar to Matlab, Wolfram Math, Maple or GeoGebra but not exatly compatible with them. We intend to use EMT in this tutorial for visual inspection of analytical solutions of ordinary differential equations.

Objectives

This is a preliminary report on the advancement of geometric interpretation of analytical solutions to ordinary differential equations. We shall

- find general solutions
- draw isoclines, direction fields, and integral curves
- identify nullclines, if any
- state existence and uniqueness theorems when applicable

We intend to show almost every step in the production of displays knowing full well that the repetitions could have been avoided. This we do to promote independent experimental verification of our examples by our Gentle Readers. (Remember, "Copy and Paste" is your friend.)

As a result, we demonstrate the relationships between direction field, isocline and integral curve; in particular we show how direction fields are tangential to integral curves and emphasize that under the given conditions integral curves fill out the domais of solution as a family of parameterized curves with no intersection or tangential contact between them.

Differential equation I.

Consider

$$x\frac{dy}{dx} + y = y^2;$$

on the Euclidean plane with standard xy coordinates.

$$\frac{dy}{dx} = \frac{y^2 - y}{x};\ \ x \neq 0;\ \ y \neq 1;\ \ y \neq 0$$

$$f(x, y) = \frac{y^2 - y}{x};\ \ f_1(x) = x;\ \ f_2(y) = y^2 - y; f(x, y) = f_2(y)/f_1(x).$$

This is a separable differential equation:

$$\frac{dy}{f_2(y)} = \frac{dx}{f_1(x)}$$

Let Q be the domain $(0.1, 20.1) \times (1.1, 21.1)$. Since $f(x, y) = \dfrac{y^2 - y}{x}$ is continuous on domain Q there exists a unique integral curve passing through each point of domain Q. First we determine solutions on Q.

$$\frac{1}{f_2(y)} = \frac{1}{y^2 - y} = \frac{1}{y - 1} - \frac{1}{y}.$$

$$\frac{dx}{x} - \frac{dy}{y - 1} + \frac{dy}{y} = 0$$

$$\int \frac{dx}{x} - \int \frac{dy}{y - 1} + \int \frac{dy}{y} = \ln |C|$$

$$\ln(x) - \ln(y - 1) + \ln(y) = \ln(C)$$

$$\frac{xy}{y - 1} = C$$

$$y = \frac{C}{C-x} \quad \text{general solution.}$$

Check:

$$\frac{dy}{dx} = \frac{-C}{(C-x)^2}; \quad y = \frac{C(C-x)}{(C-x)^2}; \quad y^2 = \frac{C^2}{(C-x)^2}$$

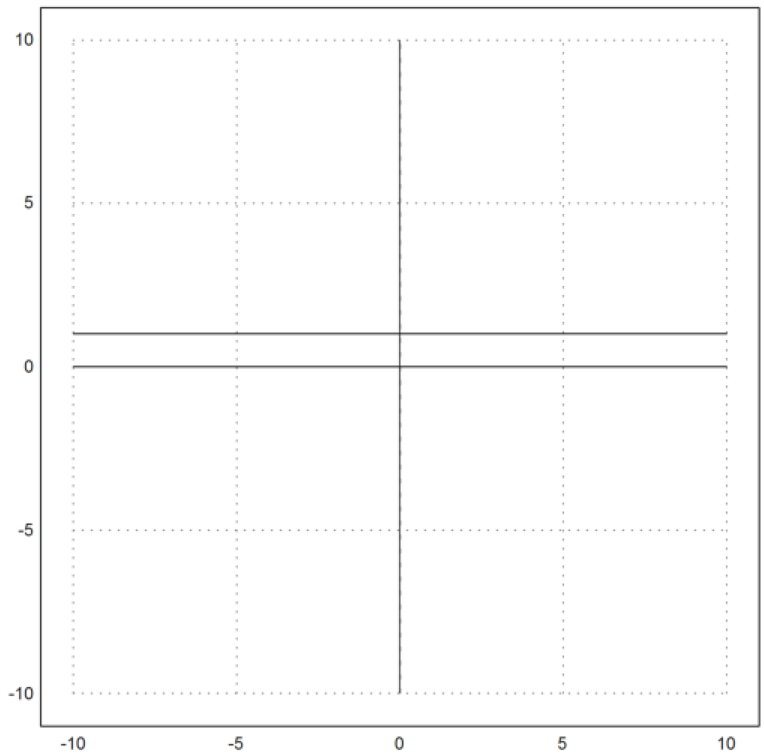
$$y^2 - y = \frac{-Cx}{(C-x)^2}.$$

$$\frac{y^2 - y}{x} = \frac{-C}{(C-x)^2} = \frac{dy}{dx}.\surd$$

$$y = 1; \; y = 0; \; \text{regular solutions} \; \surd$$

Next, we extend the solutions. Lines $x = 0$, $y = 1$, $y = 0$ partition the the Euclidean plane with standard xy coordinates. into 6 domains.

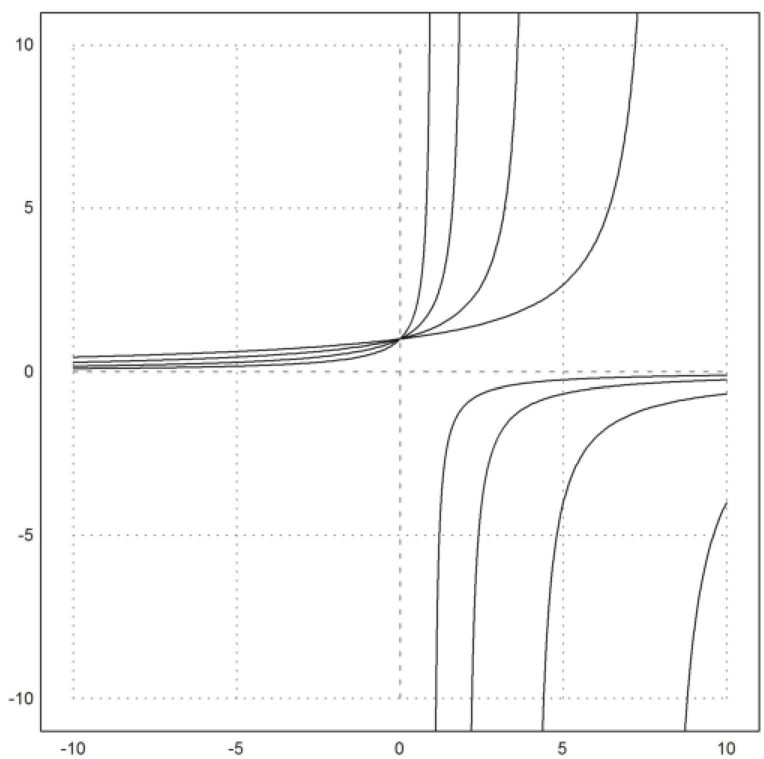
```
>function f(x,y):=x
>plot2d("f(x,y)", a=-10, b=10, c=-10, d=10, level=0):
>plot2d("0", a=-10, b=10, c=-10, d=10, >add):
>plot2d("1", a=-10, b=10, c=-10, d=10, >add):
```


10
5
0
-5
-10
-10
-5
0
5
10

Integral curves

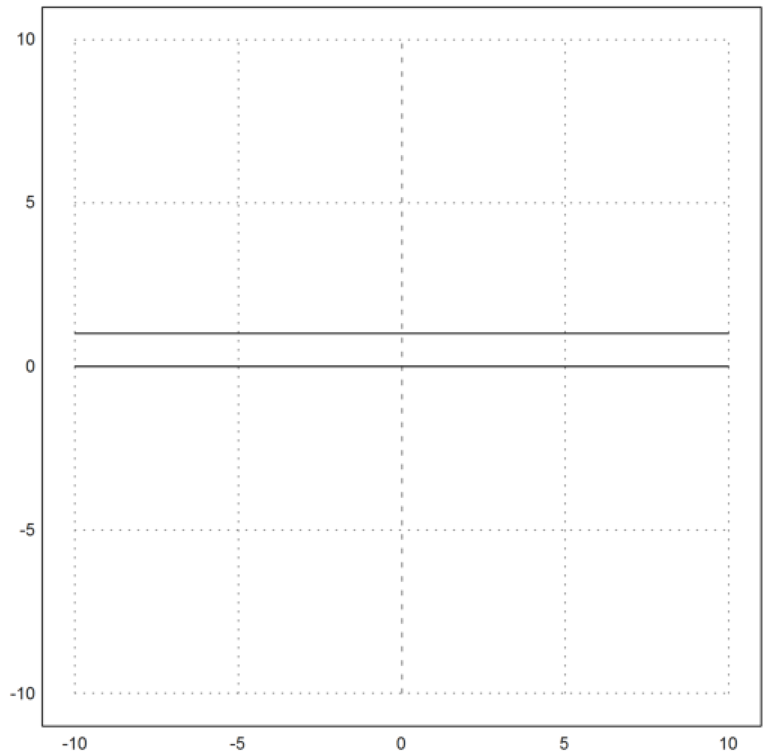
Integral curves corresponding to positive values of constants of integrations (parameter C) fill out 3 of the 6 domains. Below is a display of integral curves with $C = 1, 2, 4, 8$ on domain $(10, 10) \times (-10, 10)$: left to right $C = 1, 2, 4, 8$

```
>plot2d("1/(1-x)", a=-10, b=10, c=-10, d=10):
>plot2d("2/(2-x)", a=-10, b=10, c=-10, d=10):
>plot2d("4/(4-x)", a=-10, b=10, c=-10, d=10):
>plot2d("8/(8-x)", a=-10, b=10, c=-10, d=10):
```



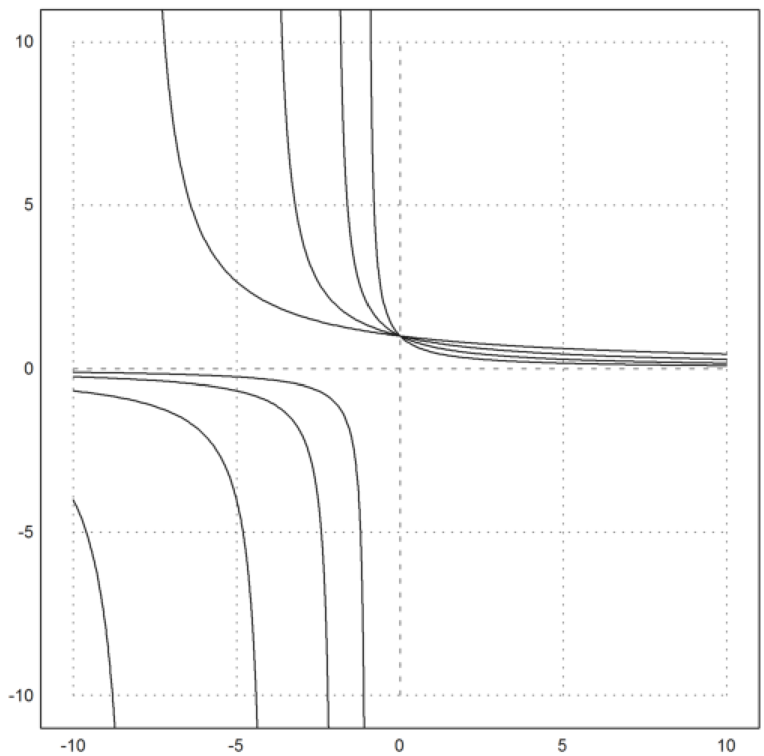
There are two regular solutions that cannot be obtained for any value of parameter C; $y = 0$ and $y = 1$.

```
>plot2d("0", a=-10, b=10, c=-10, d=10):
>plot2d("1", a=-10, b=10, c=-10, d=10, >add):
```



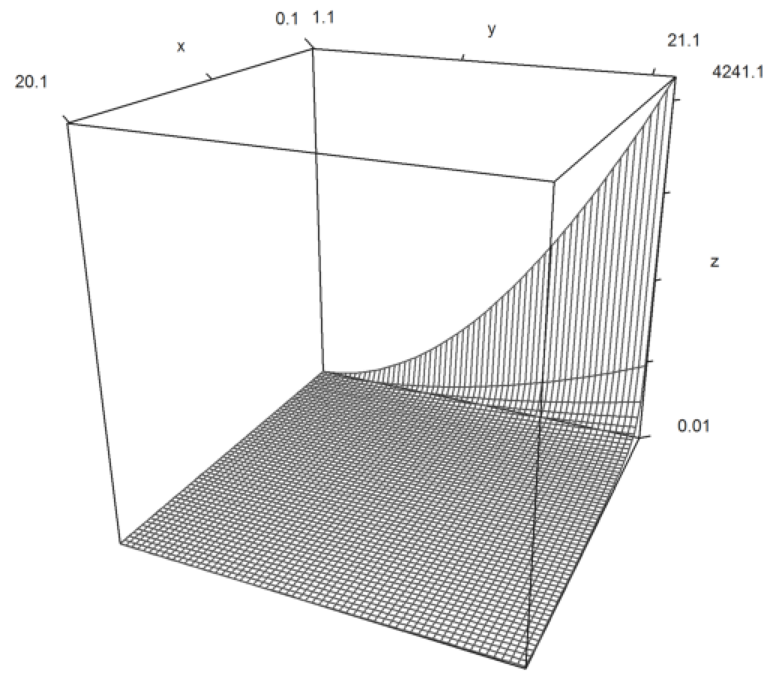
Here is a display of integral curves corresponding to $C = -1, -2, -4, -8$ on domain $(-10, 10) \times (-10, 10)$, right to left. Integral curves with negative values of constants of integrations (parameter C) fill out the other 3 domains.

```
>plot2d("-1/(-1-x)", a=-10, b=10, c=-10, d=10):
>plot2d("-2/(-2-x)", a=-10, b=10, c=-10, d=10):
>plot2d("-4/(-4-x)", a=-10, b=10, c=-10, d=10):
>plot2d("-8/(-8-x)", a=-10, b=10, c=-10, d=10):
```



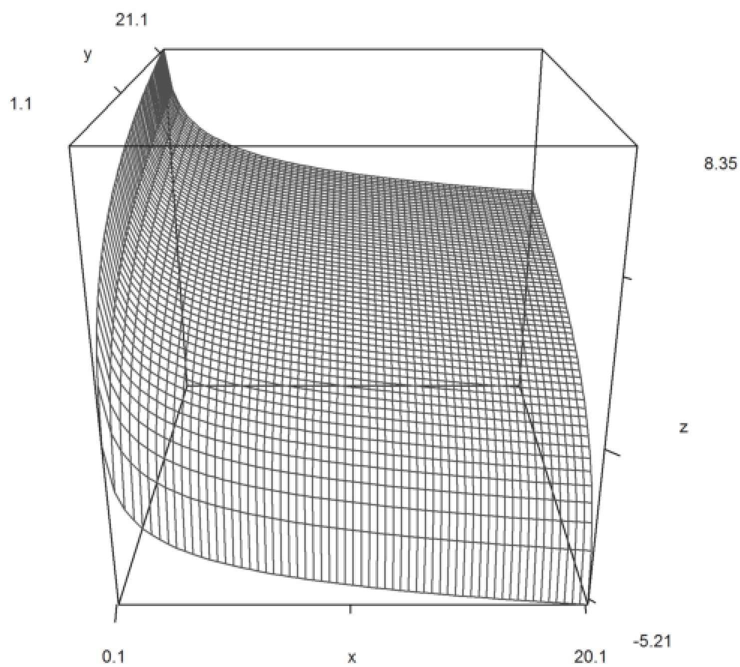
Wire diagram of f(x,y)

```
>function f(x,y):= (y^2-y)/x
>plot3d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, >wire);
```



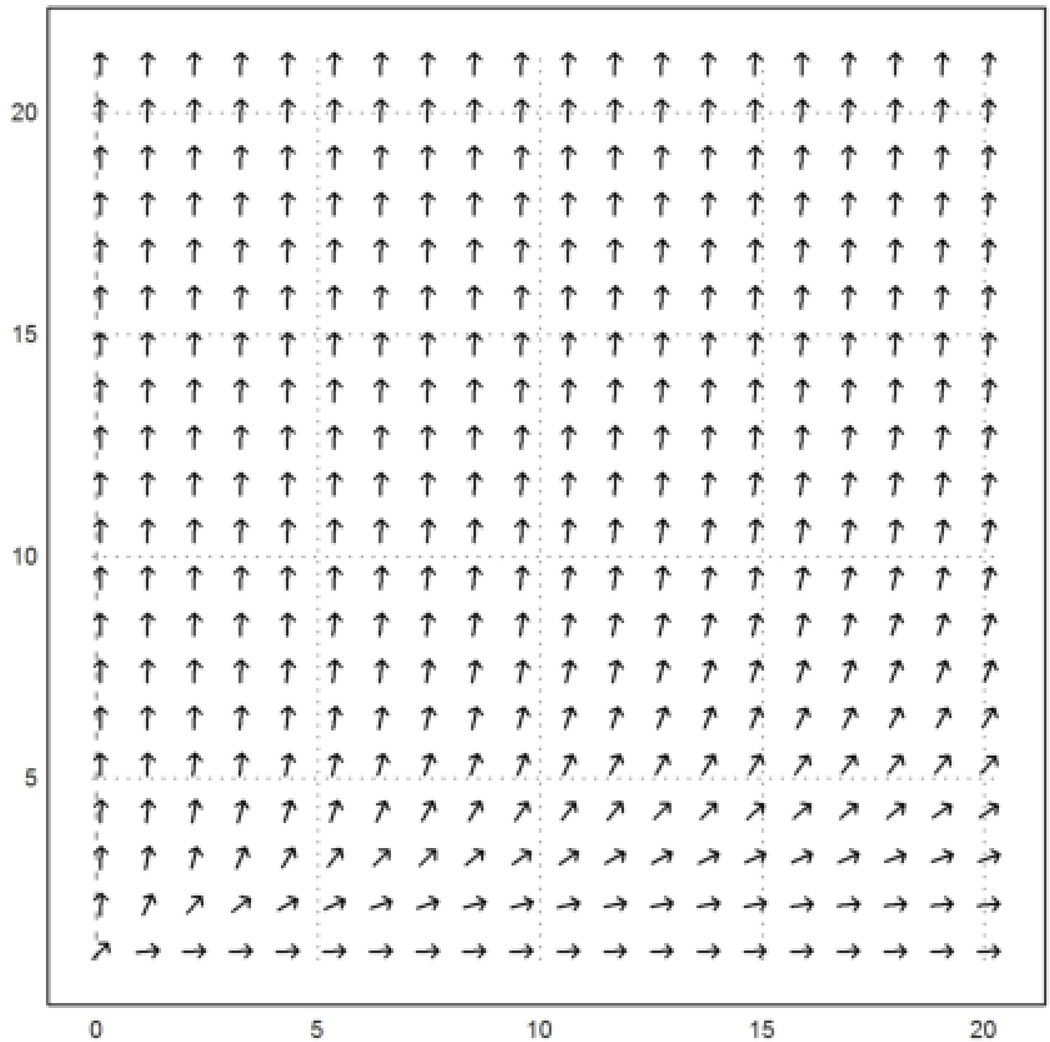
Stretched Wire diagram of g(x,y)=log(f(x,y))

```
>function g(x,y):= log((y^2-y)/x)
>plot3d("g(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, >wire, angle=45 );
```


21.1
y
1.1
8.35
z
0.1
x
20.1
-5.21

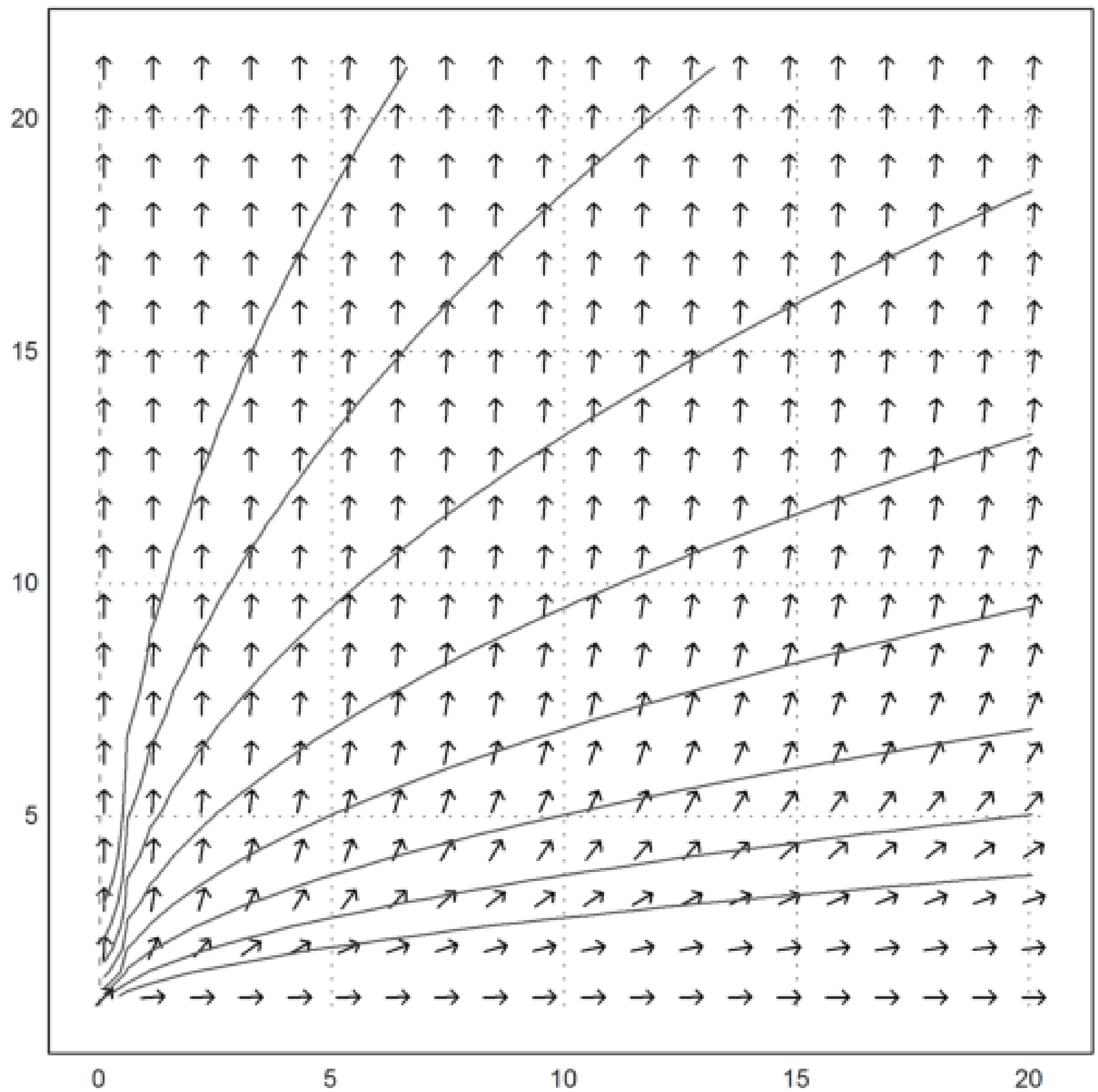
Direction field of (f(x,y)

```
>vectorfield("f(x,y)", x1=0.1, x2=20.1, y1=1.1, y2=21.1 ):
```



Isoclines of f(x,y) on direction field

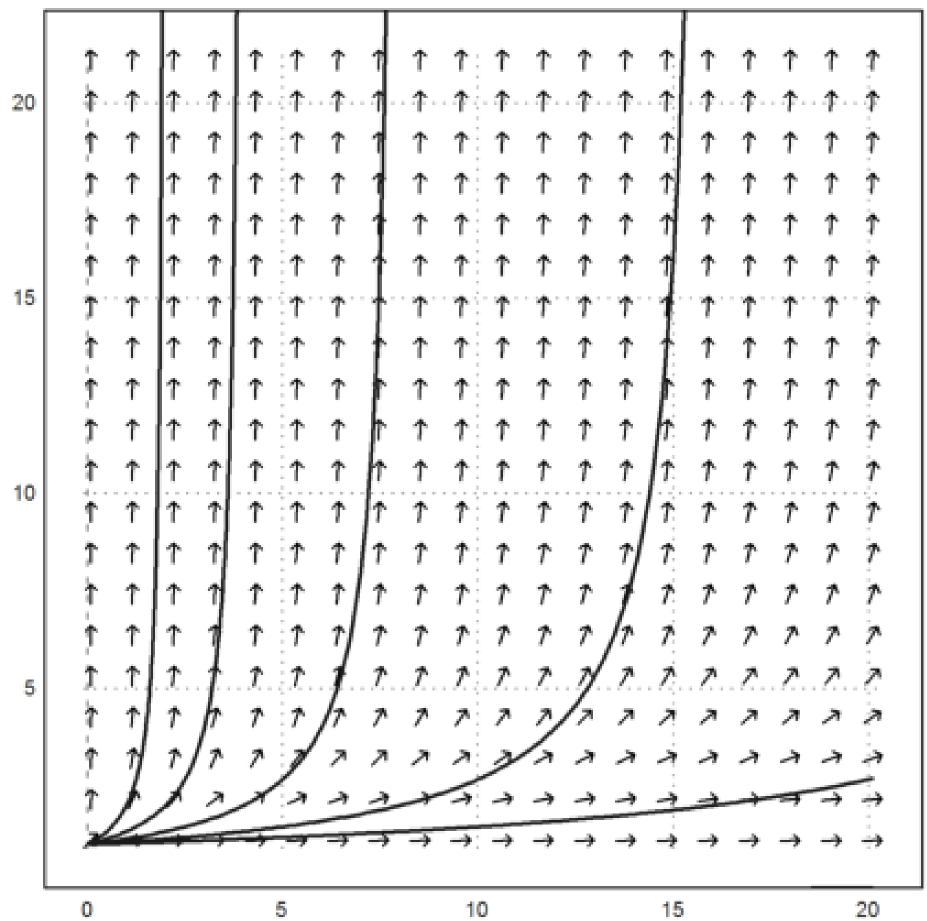
```
>plot2d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, level=0.5, >add):
>plot2d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, level=1, >add):
>plot2d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, level=2, >add):
>plot2d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, level=4, >add):
>plot2d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, level=8, >add):
>plot2d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, level=16, >add):
>plot2d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, level=32, >add):
>plot2d("f(x,y)", a=0.1, b=20.1, c=1.1, d=21.1, level=64, >add):
```



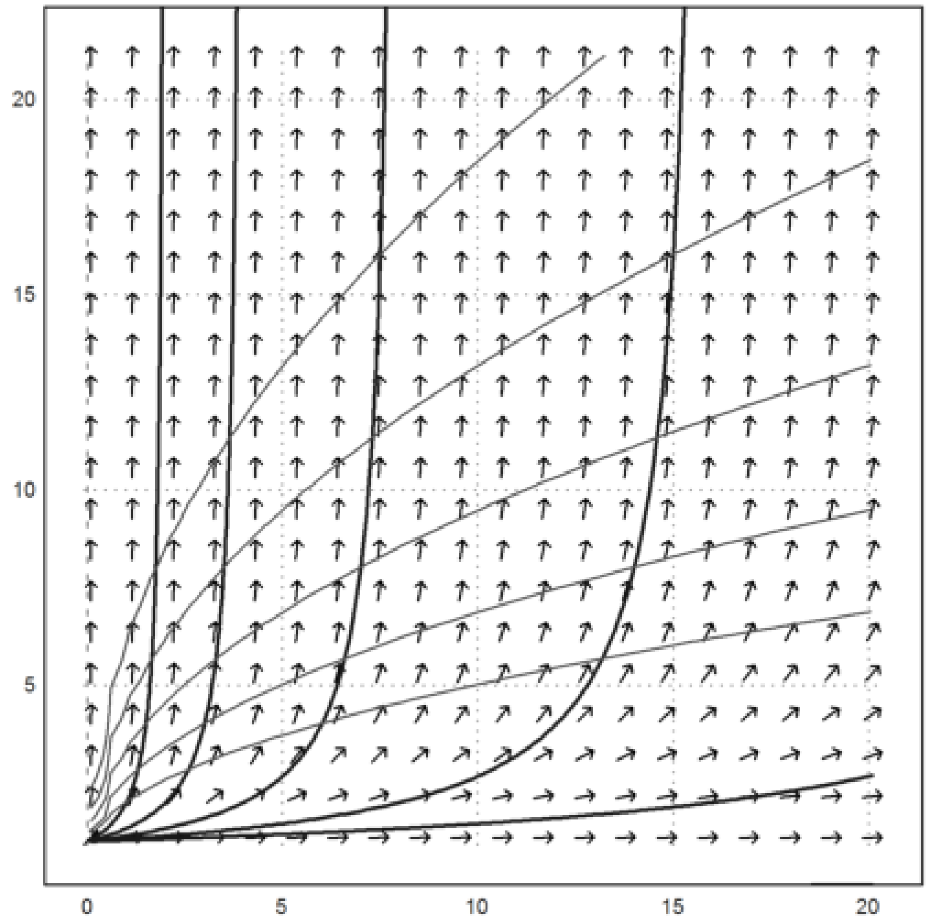
20
15
10
5
0
5
10
15
20

Integral curves on direction field

```
>plot2d("0.5/(x+0.5)", a=0.1, b=20.1, c=1.1, d=21.1, thickness=2,>add):
>plot2d("1/(x+1)", a=0.1, b=20.1, c=1.1, d=21.1, thickness=2, >add):
>plot2d("0/(x+0)", a=0.1, b=20.1, c=1.1, d=21.1, thickness=2, >add):
>plot2d("-13/(x-13)", a=0.1, b=20.1, c=1.1, d=21.1, thickness=2, >add):
>plot2d("-26/(x-26)", a=0.1, b=20.1, c=1.1, d=21.1, thickness=2, >add):
```



Composite diagram, result



Differential Equation II.

$$(y-2)dx-(x+1)dy=0$$

$$(y-2)dx=(x+1)dy$$

$$\frac{dy}{dx}=\frac{y-2}{x+1}=f(x,y)$$

The Euclidean plane with standard xy coordinates is partitioned into 4 domains by $y=2$ and $x=-1$. We solve and display the above differential equation on the patch $(-0.1, 19.1)\times(2.1, 22.1)$ where $f(x,y)$ is smooth, ($f(x,y)$ is continuous, and $f(x,y)$ is countinuously differentiable with respect to y) the ode is separable, hence has a unique solution.

$$\frac{dy}{y-2}=\frac{dx}{x+1}$$

$$\frac{dy}{y-2}-\frac{dx}{x+1}=0$$

$$\int\frac{dy}{y-2}-\int\frac{dx}{x+1}=C$$

$$\ln{(y-2)}-\ln{(x+1)}=\ln C$$

$$\frac{y-2}{x+1}=C$$

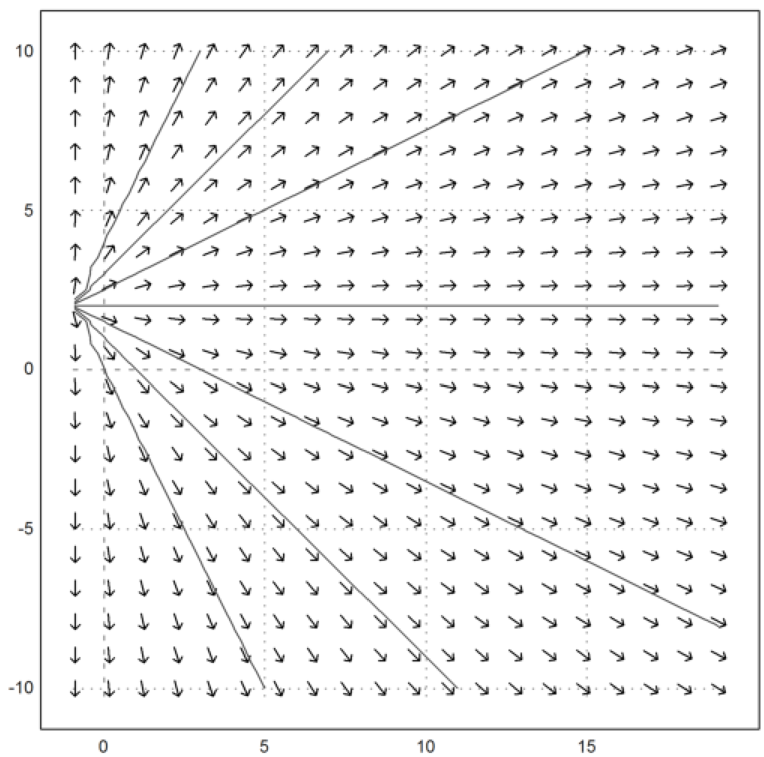
$$y-2=C(x+1);\;\; x\neq -1, y\neq 2$$

Upon calculating isocline $f(x,y)=C$ we obtain

$$\frac{y-2}{x+1}=C$$

as above, so so isoclines and integral curves coincide and direction field is tangential to solution, as it should.

```
>function f(x,y):= (y-2)/(x+1)
>vectorfield("f(x,y)", x1=-0.9, x2=19.1, y1=-10, y2=10):
>plot2d("f(x,y)", a=-0.9, b=19.1, c=-10, d=10, level=-2, >add):
>plot2d("f(x,y)", a=-0.9, b=19.1, c=-10, d=10, level=-1, >add):
>plot2d("f(x,y)", a=-0.9, b=19.1, c=-10, d=10, level=-0.5, >add):
>plot2d("f(x,y)", a=-0.9, b=19.1, c=-10, d=10, level=0, >add):
>plot2d("f(x,y)", a=-0.9, b=19.1, c=-10, d=10, level=0.5, >add):
>plot2d("f(x,y)", a=-0.9, b=19.1, c=-10, d=10, level=1, >add):
>plot2d("f(x,y)", a=-0.9, b=19.1, c=-10, d=10, level=2, >add):
```



20.1 Tutorial 17.

- Ordinary Differential equations
- *Spartan Old School*
- Last revision January 15, 2020

Pencils of curves and equations

In this tutorial we sample some familiar pencils of curves and find the ordinary differential equations that defines them. The EMT instructions are under **Snippets.**

Preliminaries The general solution of the first order ordinary differential equation

$$F(x, y, y\prime) = 0$$

has one constant of integration C

$$\Phi(x, y, C) = 0$$

where Φ is a one-parameter pencil (or family) of curves.

The general solution of the second order ordinary differential equation

$$F(x, y, y\prime, y\prime\prime) = 0$$

has two constants of integration C_1, C_2

$$\Phi(x, y, C_1, C_2) = 0.$$

The $n - th$ order ordinary differential equation

$$F(x, y, y\prime, y\prime\prime, \ldots, y^{(n)}) = 0$$

has a general solution with n parameters

$$\Phi(x, y, C_1, C_2, \ldots, C_n) = 0.$$

Generally, if an n-parameter pencil of curves is given we can find the differential equation that generates it :

$$\Phi(x, y, C_1, C_2, \ldots, C_n) = 0 \Rightarrow F(x, y, y\prime, y\prime\prime, \ldots, y^{(n)}) = 0.$$

First, we differentiate the equation $\Phi = 0$ n-times:

$$\begin{aligned} \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} y\prime &= 0 \\ \frac{\partial^2 \Phi}{\partial x^2} + 2\frac{\partial^2 \Phi}{\partial x \partial y} y\prime + \frac{\partial^2 \Phi}{\partial y^2} y\prime^2 + \frac{\partial \Phi}{\partial y} y\prime\prime &= 0 \\ \frac{\partial^3 \Phi}{\partial x^3} + 2\frac{\partial^3 \Phi}{\partial x^2 \partial y} y\prime + \ldots + \frac{\partial \Phi}{\partial y} y\prime\prime\prime &= 0 \\ \cdots\cdots\cdots \\ \frac{\partial^n \Phi}{\partial x^n} + \ldots\ldots + \frac{\partial \Phi}{\partial y} y^{(n)} &= 0 \end{aligned}$$

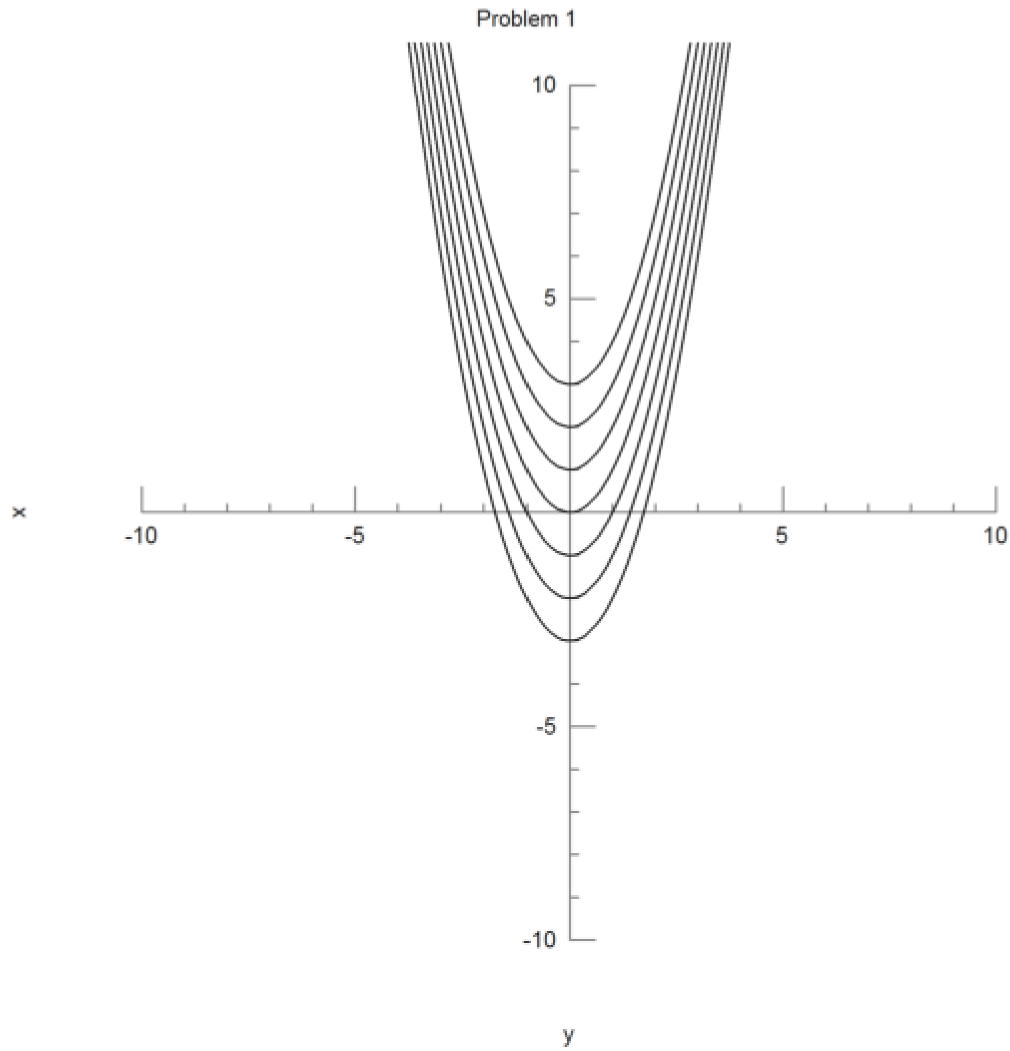
Next, we eliminate $C_1, C_2, \ldots C_n$ from the above system of $n + 1$ equations, and obtain $F = 0$ if certain conditions are met.

1. Parabolas I.

$y = x^2 + C.$

$y\prime = 2x$

$C = -3, -2, -1, 0, 1, 2, 3$


Problem 1
10
5
-5
-10
x
-10
-5
5
10
y

2. Parabolas II.

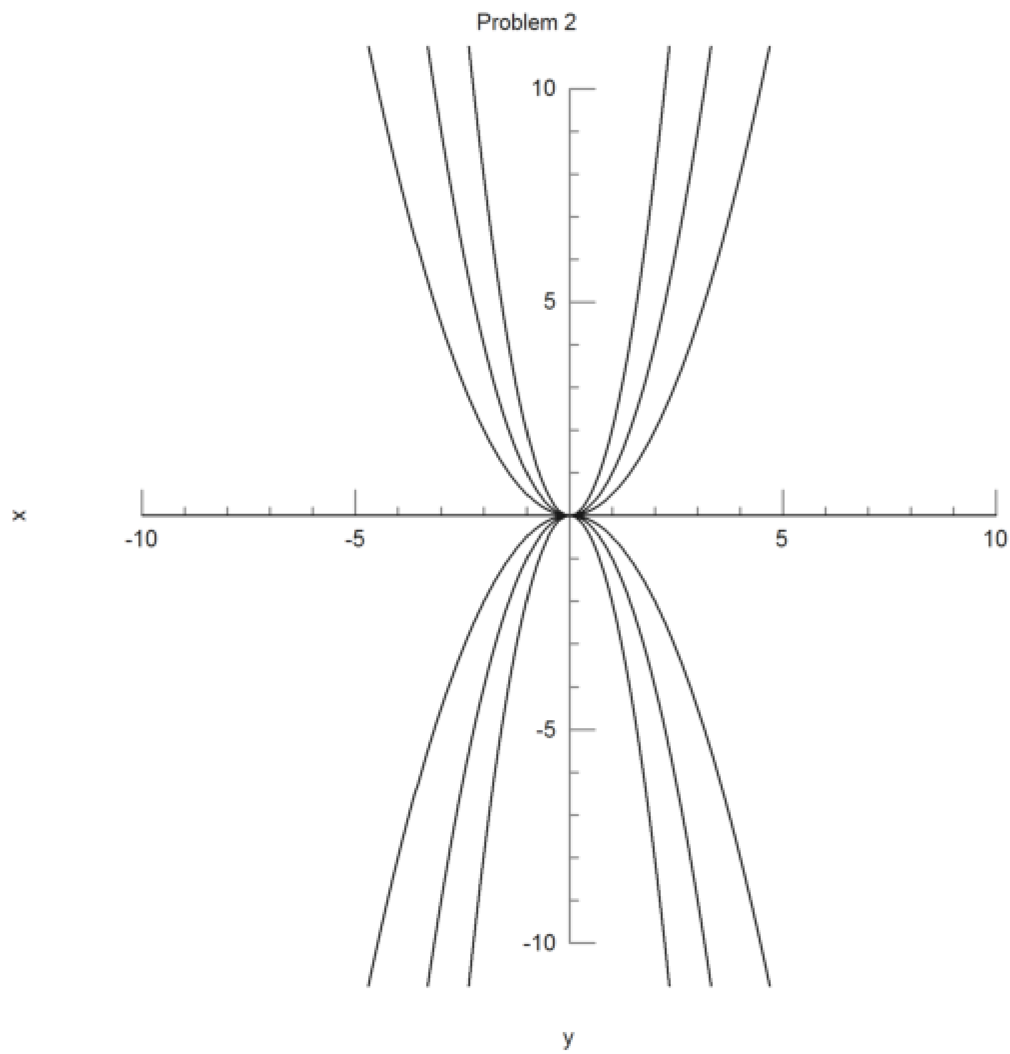
$y = Cx^2$

$y\prime = 2Cx$

$\frac{y\prime}{2x} = C$

$y\prime = 2\frac{y}{x}$

$C = -2, -1, -0.5, 0, 0.5, 1, 2$



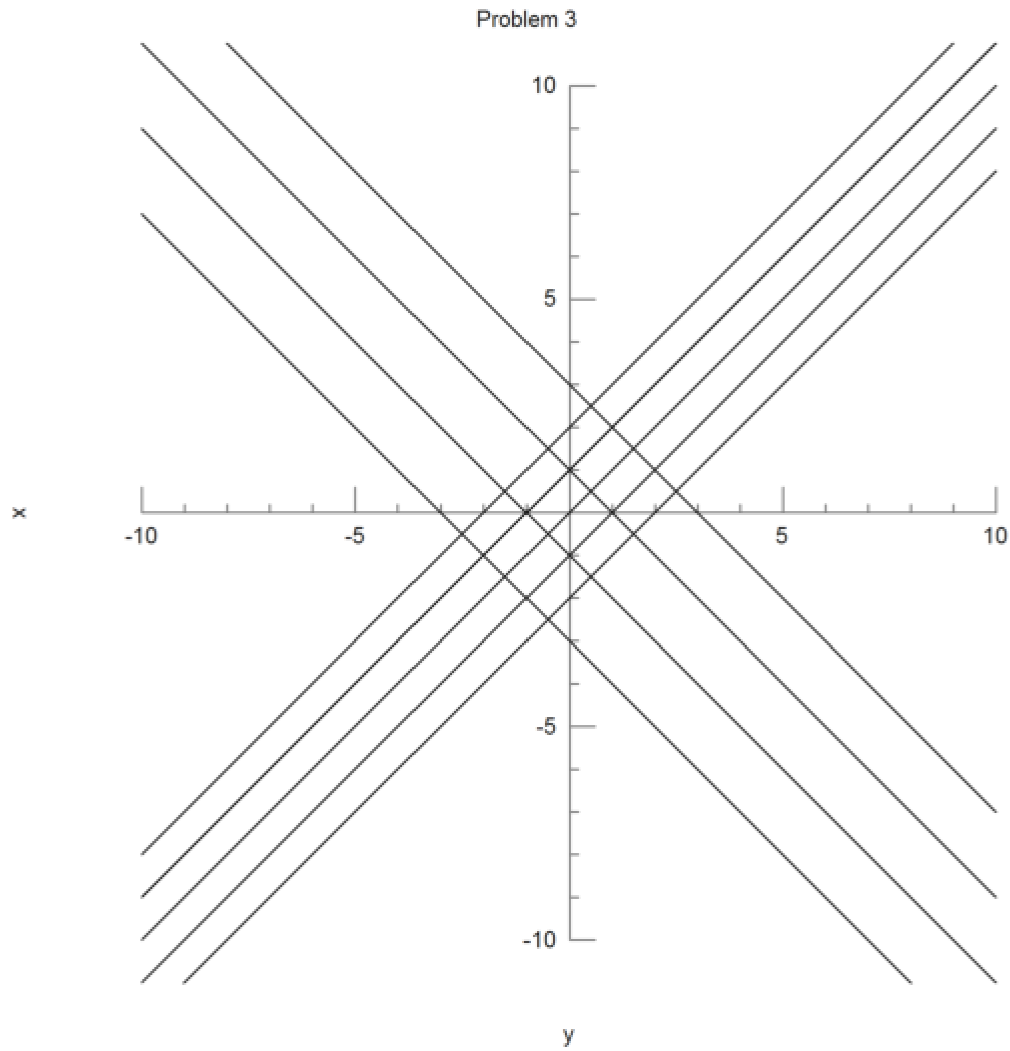
3. Lines I.

$$y = C_1 x + C_2$$

$$y\prime = C_1$$

$$y\prime\prime = 0$$

$$C_1 = \pm 1, C_2 = -3, -2, ..2, 1$$



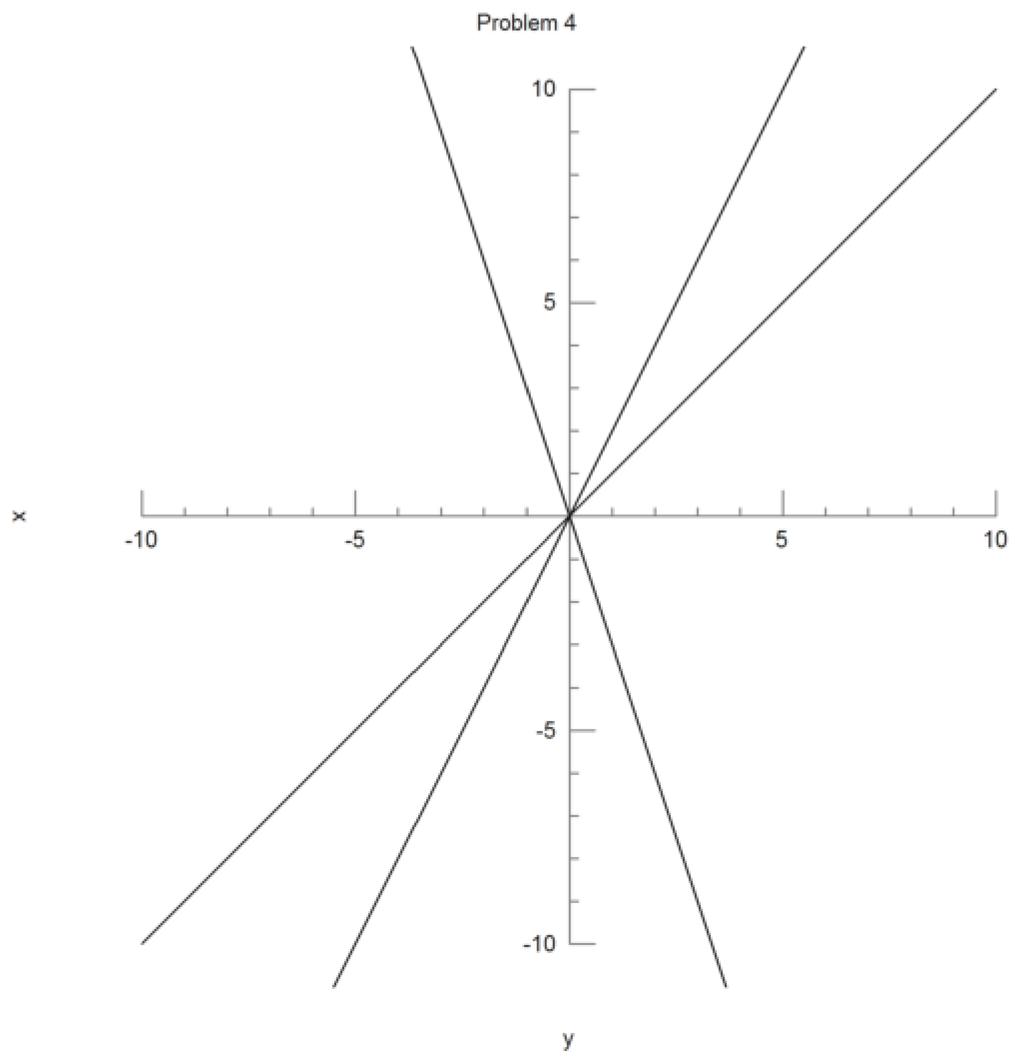
4. Lines II.

$$y = Cx$$

$$y\prime = C$$

$$y\prime = \frac{y}{x}$$

$$C = 1, 2, -3$$



5. Circles I.

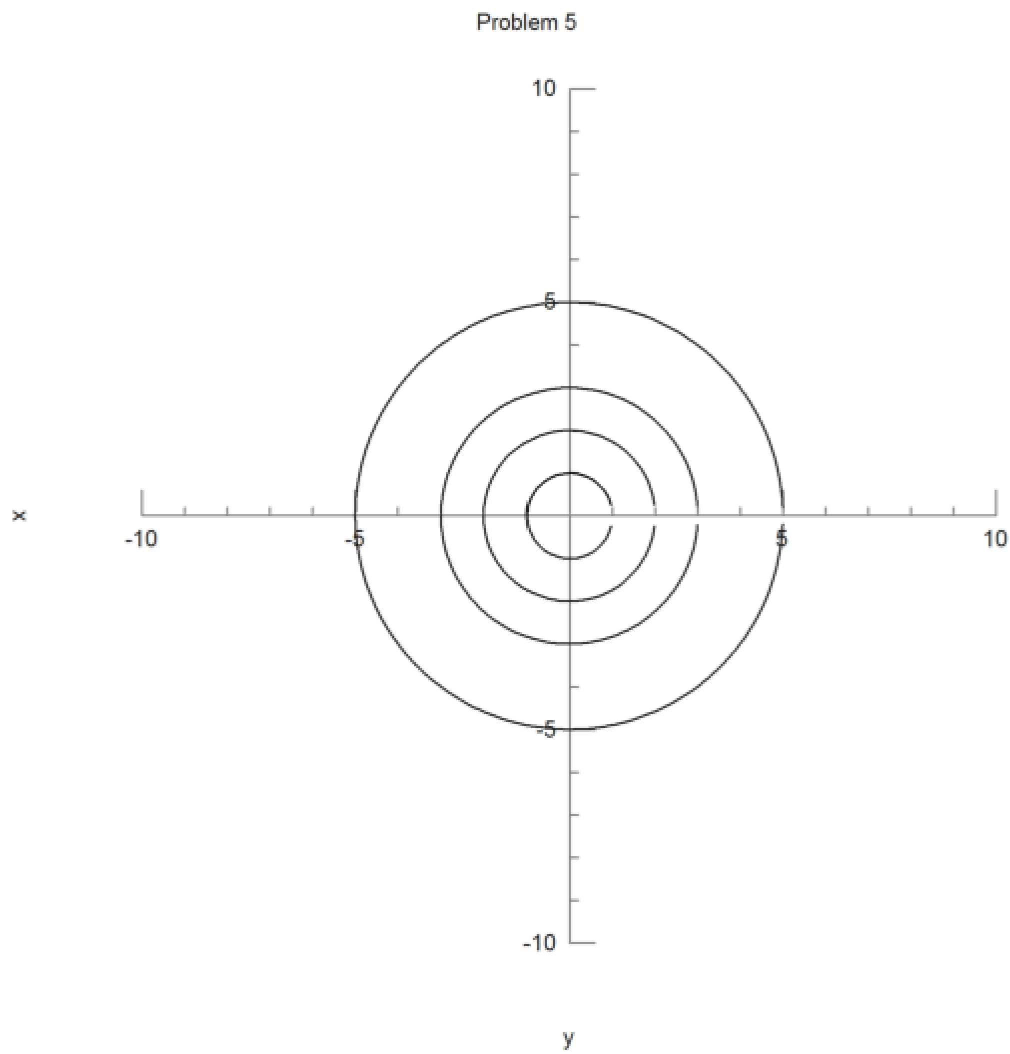
$$x^2 + y^2 = C$$

$$2x + 2yy\prime = 0$$

$$y\prime = -\frac{x}{y}$$

$$x + yy\prime = 0$$

$$C = 1, 2, 3, 5$$



6. Circles II.

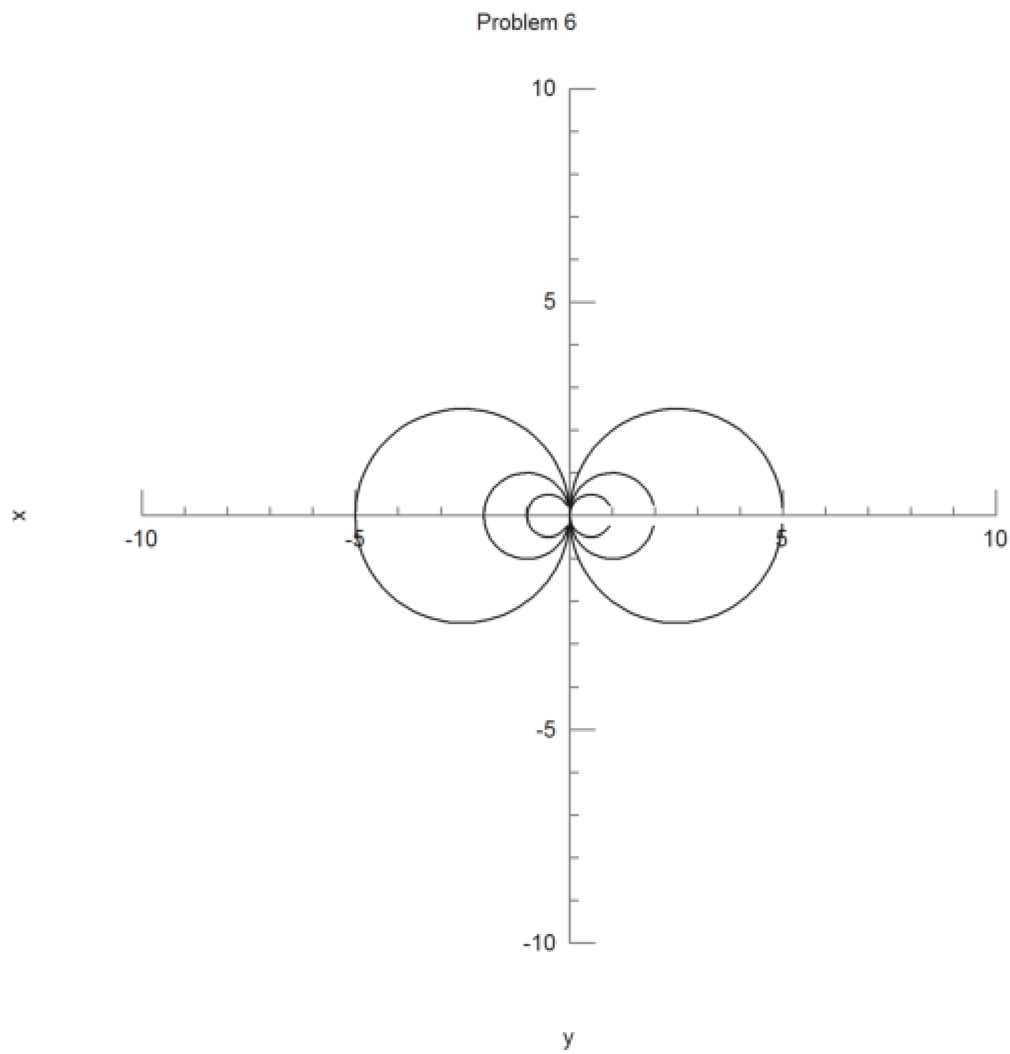
$$x^2 + y^2 = Cx$$

$$2x + 2yy\prime = C$$

$$x^2 + y^2 = 2(x + yy\prime)x$$

$$x^2 - y^2 + 2xyy\prime = 0$$

$$C = \pm 1, \pm 2, \pm 5$$



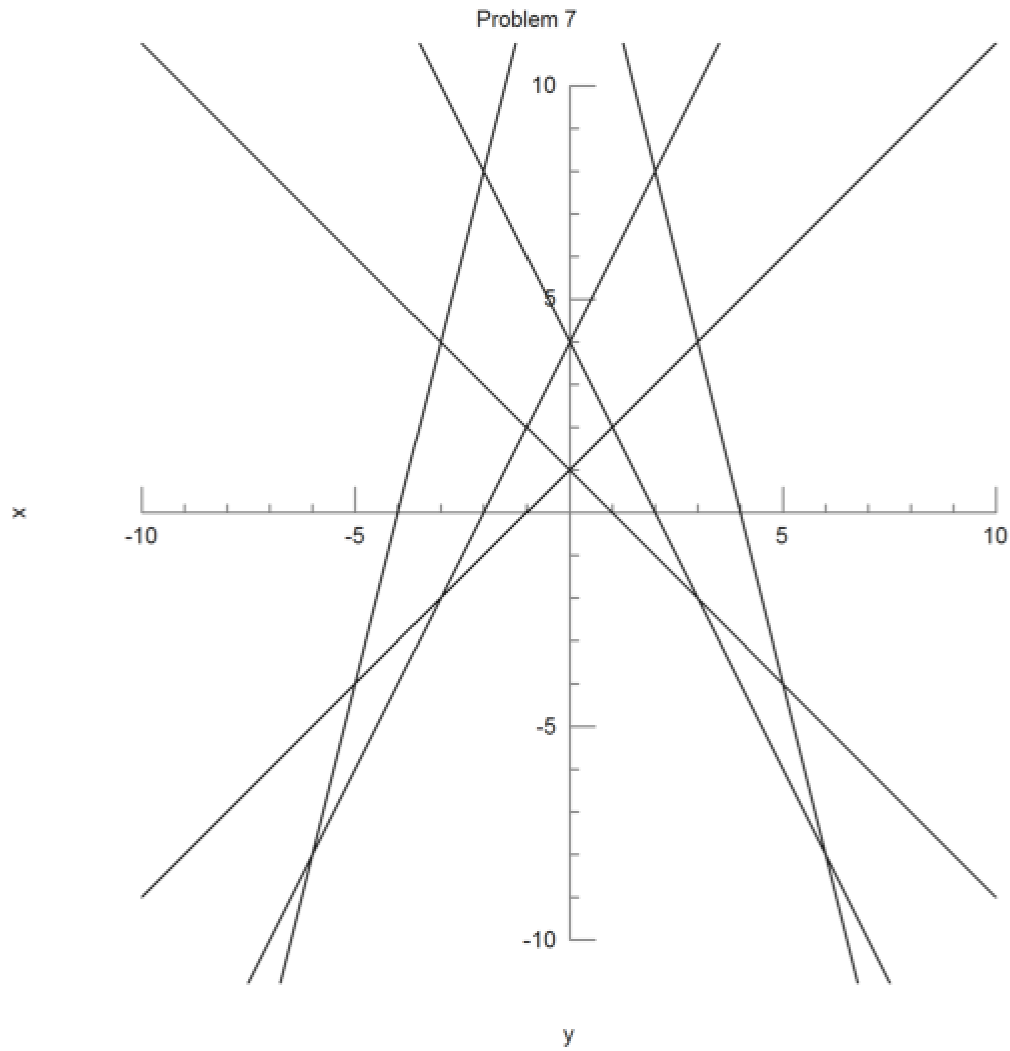
7. Lines III.

$$y = Cx + C^2$$

$$y\prime = C$$

$$y = xy\prime + y\prime^2$$

$$C = \pm 1, \pm 2, \pm 4$$



8. Lemniscate of Bernoulli.

$$(x^2+y^2)^2 = C^2(x^2-y^2)$$

$$2(x^2+y^2)(2x+2yy\prime) = C^2(2x-2yy\prime)$$

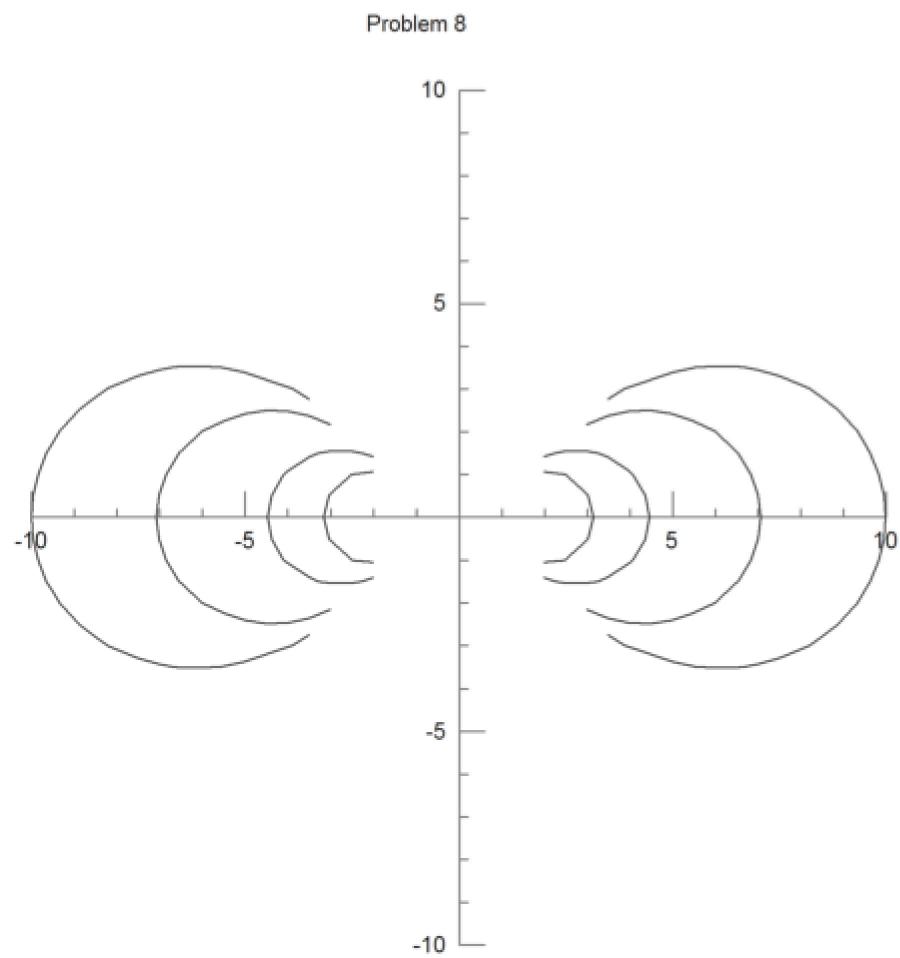
$$C^2 = \frac{2(x^2+y^2)(2x+2yy\prime)}{(2x-2yy\prime)}$$

$$(x^2+y^2)^2 = \frac{2(x^2+y^2)(2x+2yy\prime)}{(2x-2yy\prime)}(x^2-y^2)$$

$$(x^2+y^2) = \frac{2(2x+2yy\prime)}{(2x-2yy\prime)}(x^2-y^2)$$

$$\frac{x-yy\prime}{x^2-y^2} = 2\frac{x+yy\prime}{x^2+y^2}.$$

$$C^2 = 10, 20, 50, 100$$



9. Exponential curves.

$$y = C \exp\left(\frac{x}{C}\right)$$

$$y\prime = \frac{1}{C} C \exp\left(\frac{x}{C}\right) = \exp\left(\frac{x}{C}\right)$$

$$\ln y = \ln C + \frac{x}{C}$$

$$\ln y\prime = \frac{x}{C}$$

From $y = C \exp\left(\frac{x}{C}\right)$

$$C = \frac{y}{y\prime},$$

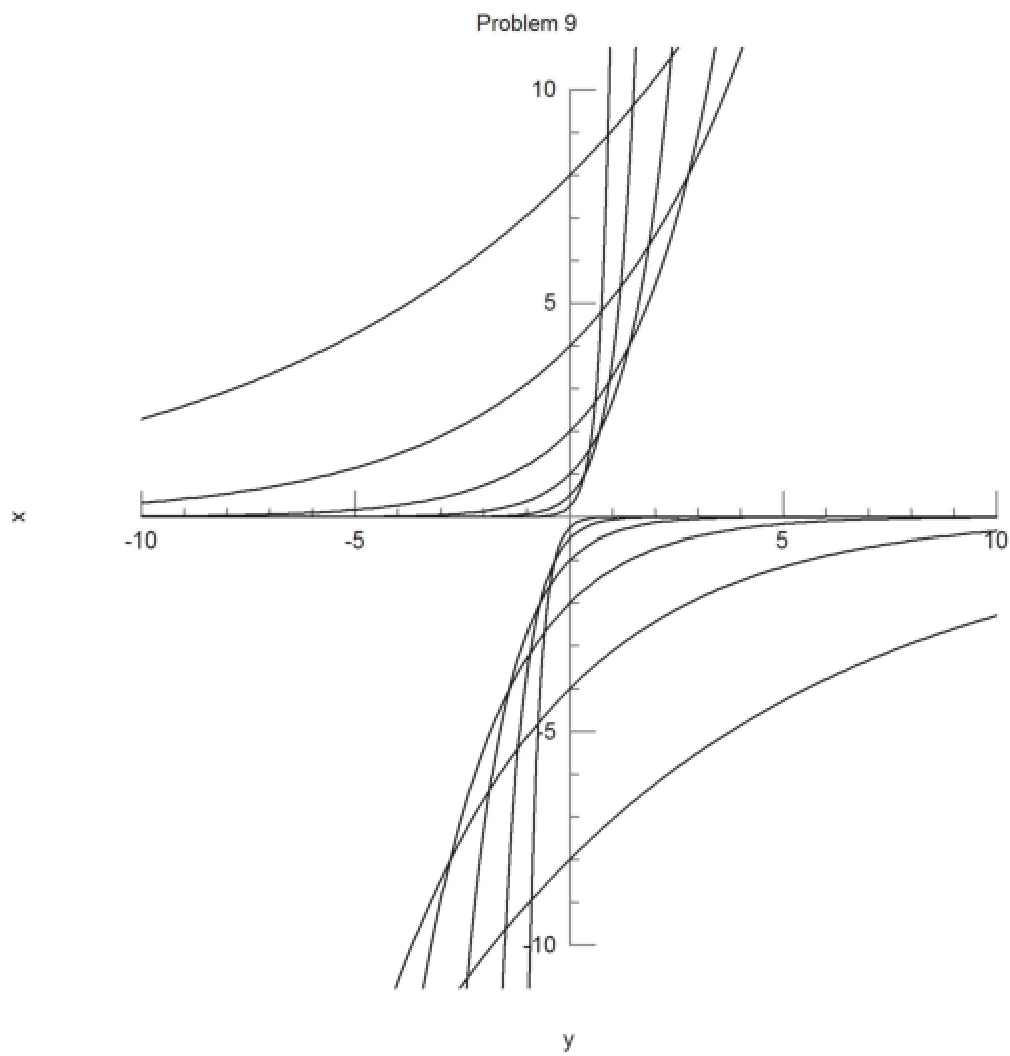
moreover

$$xy\prime = x \exp\left(\frac{x}{C}\right)$$

$$xy\prime = \frac{Cx}{C} \exp\left(\frac{x}{C}\right) = \frac{x}{C} C \exp\left(\frac{x}{C}\right)$$

$$xy\prime = y \ln y\prime.$$

$$C = \pm 0.25, \pm 0.5, \pm 1, \pm 2, \pm 4, \pm 8$$



10. Sinusoid curves.

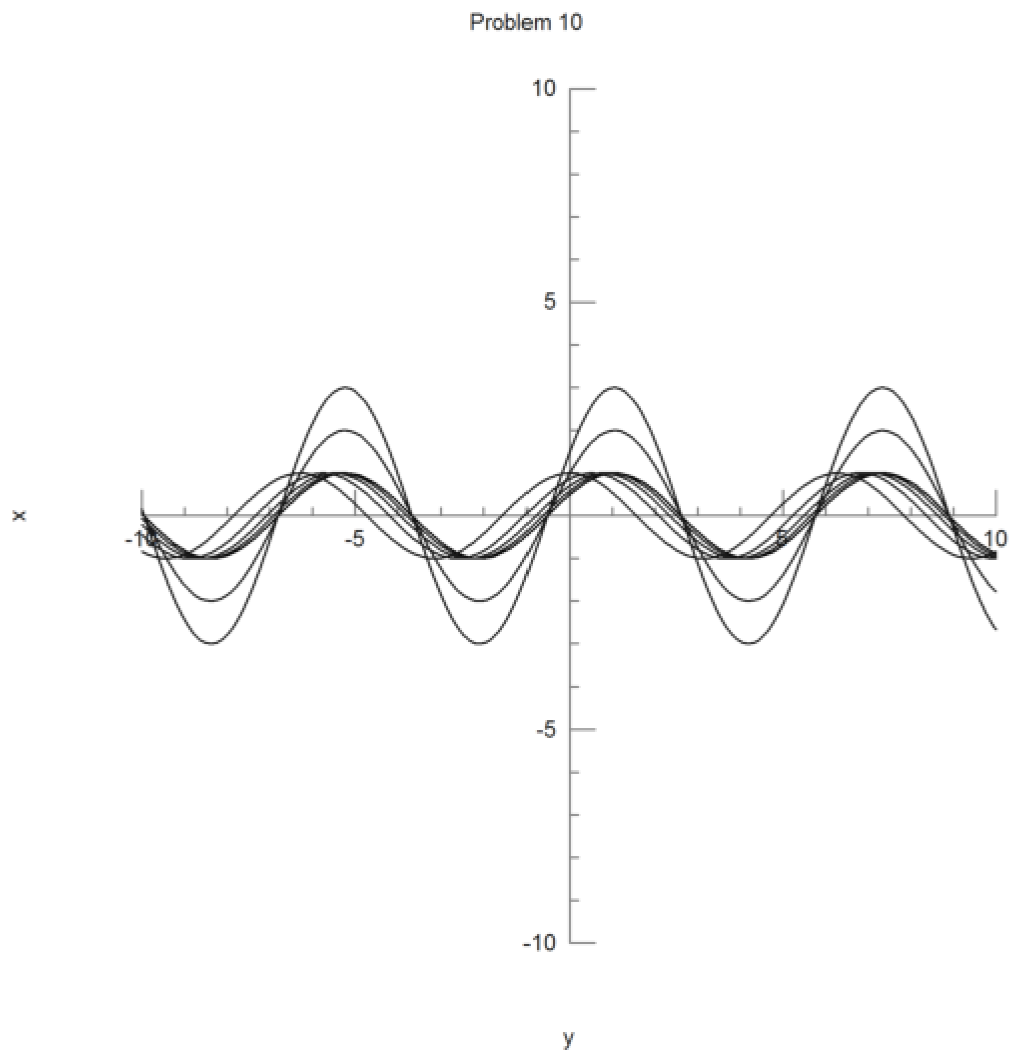
$$y = C_1 \sin(x + C_2)$$

$$y\prime = C_1 \cos(x + C_2)$$

$$y\prime\prime = -C_1 \sin(x + C_2) = -y$$

$$y\prime\prime + y = 0.$$

$$C_1 = 1, 2, 3, C_2 = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}$$



Snippets

1. Parabolas I.

```
>plot2d("x", "x^2+1", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title="Problem
>plot2d("x", "x^2+2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "x^2+3", a=-10, b=10, c=-10, d=10, >add)
```

```
>plot2d("x", "x^2+0", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "x^2-1", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "x^2-2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "x^2-3", a=-10, b=10, c=-10, d=10, >add)
```

2. Parabolas II

```
>plot2d("x", "1*x^2", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title="Problem
>plot2d("x", "2*x^2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "0.5*x^2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "0.0*x^2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "-1*x^2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "-2*x^2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "-0.5*x^2", a=-10, b=10, c=-10, d=10, >add)
```

3. Lines I

```
>plot2d("x","1*x+1", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title="Problem 3
>plot2d("x","1*x+2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x","1*x+0", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x","1*x-1", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x","1*x-2", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x","(-1)*x+1", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x","(-1)*x+3", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x","(-1)*x-1", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x","(-1)*x-3", a=-10, b=10, c=-10, d=10, >add)
```

4. Lines II

```
>plot2d("x", "1*x", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title="Problem 4'
>plot2d("x", "2*x", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "-3*x", a=-10, b=10, c=-10, d=10, >add)
```

5. **Circles I.**

```
>plot2d("x", "sqrt(1^2-x^2)", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title='
>plot2d("x", "-sqrt(1^2-x^2)" , >add)
>plot2d("x", "-sqrt(2^2-x^2)" , >add); plot2d("x", "-sqrt(2^2-x^2)" , >add)
>plot2d("x", " sqrt(3^2-x^2)" , >add); plot2d("x", "-sqrt(3^2-x^2)" , >add)
>plot2d("x", "sqrt(2^2-x^2)" , >add)
>plot2d("x", " sqrt(5^2-x^2)" , >add); plot2d("x", "-sqrt(5^2-x^2)" , >add)
```

6. **Circles II.**

```
>plot2d("x", "sqrt(1*x-x^2)", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title='
>plot2d("x", "sqrt(2*x-x^2)", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "sqrt(5*x-x^2)", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "sqrt((-1)*x-x^2)", a=-10, b=10, c=-10, d=10, >add)
>plot2d("x", "sqrt((-2)*x-x^2)", a=-10, b=10, c=-10, d=10, >add) ...
```

7. **Lines III.**

```
>plot2d("x","1*x+1^2", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title="Problem
>plot2d("x","2*x+2^2", >add)
>plot2d("x","4*x+4^2", >add)
>plot2d("x","(-1)*x+(-1)^2", >add)
>plot2d("x","(-2)*x+(-2)^2", >add)
>plot2d("x","(-4)*x+(-4)^2", >add)
```

8. **Lemniscate of Bernoulli.**

```
> function f(x,y):= (x^2+y^2)^2/(x^2-y^2)
> plot2d( "f(x,y)", a=-10, b=10, c=-10, d=10, level=20,  xl="y", yl="x", grid=8
> plot2d( "f(x,y)", a=-10, b=10, c=-10, d=10, level=20,  xl="y", yl="x", title=
> plot2d( "f(x,y)", level=10, >add)
> plot2d( "f(x,y)", level=50, >add)
> plot2d( "f(x,y)", level=100, >add)
```

9. Exponential curves.

```
>plot2d("x","1*exp(x/1)", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title="Prob
>plot2d("x","2*exp(x/2)", >add)
>plot2d("x","4*exp(x/4)", >add)
>plot2d("x","8*exp(x/8)", >add) ...
```

10. Sinusoid curves.

```
>plot2d("x", "1*sin(x+(pi/6))", a=-10, b=10, c=-10, d=10, xl="y", yl="x", title
>plot2d("x", "2*sin(x+(pi/6))", >add)
>plot2d("x", "3*sin(x+(pi/6))", >add)
>plot2d("x", "1*sin(x+(pi/5))", >add)
>plot2d("x", "1*sin(x+(pi/4))", >add)
>plot2d("x", "1*sin(x+(pi/3))", >add)
>plot2d("x", "1*sin(x+(pi/2))", >add)
```

20.2 Assignment 41.

Summary

- Mathematical Modelling and Numerical Analysis, Morton-Mayers
- Separation of variables, Fourier series
- Last revision January 15, 2020

Heat equation, model problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \quad x\varepsilon[0,1], \quad t \geq 0, \tag{20.1}$$

$$u(0,t) = u(1,t) = 0, \tag{20.2}$$

$$u(x,0) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} \leq x \leq 1. \end{cases} \tag{20.3}$$

Separation of variables

Assume

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{\partial u}{\partial t} = X(x)T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

Therefore

$$X(x)T'(t) = X''(x)T(t)$$

or

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

This implies

$$\frac{T'(t)}{T(t)} = \frac{X"(x)}{X(x)} = -\lambda^2,$$

and we have two separate ordinary differential equations. The first one is for $X(x)$:

$$-X"(x) = \lambda^2 X(x) \tag{20.4}$$

$$X(0) = X(1) = 0 \tag{20.5}$$

It is a Sturm-Liouville problem with eigenvalues

$$\lambda_n^2 = (n\pi)^2$$

and corresponding eigenfunctions

$$X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

The second one is

$$T'(t) = -\lambda^2 T(t).$$

A solution is

$$T_n(t) = \ exp(-\lambda_n^2 t) = \ exp(-(n\pi)^2 t) \ \ t \geq 0, \quad n = 1, 2, \ldots$$

(The choice of $-\lambda^2$ is justified because the solution does not grow out of bounds.) Therefore we claim that each

$$u_n(x,t) = X_n(x)T_n(t) = \sin(n\pi x) \ exp(-(n\pi)^2 t) \tag{20.6}$$

satisfies heat equation (20.1), and boundary conditions (20.2):

$$\frac{\partial u_n(x,t)}{\partial t} = -(n\pi)^2 \sin(n\pi x) \ exp(-(n\pi)^2 t)$$

$$\frac{\partial^2 u_n(x,t)}{\partial x^2} = -(n\pi)^2 \sin(n\pi x) \ exp(-(n\pi)^2 t)$$

$$u_n(0,t) = \sin(n\pi 0) \ exp(-(n\pi)^2 t) = 0$$

$$u_n(1,t) = \sin(n\pi) \ exp(-(n\pi)^2 t) = 0.$$

Superposition, Fourier series

Next, we develop the Fourier series approximation of the "hat function" which is the initial condition for the model problem.

(i) The function $u^0(x)$ is defined on $[0,1]$ by

$$u^0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \tag{20.7}$$

Show that

$$u^0(x) = \sum_{m=1}^{\infty} a_m \sin(m\pi x)$$

where

$$a_m = (8/m^2\pi^2)\sin(\frac{1}{2}m\pi).$$

(Morton - Mayers, Ex. 2.1)

Proof: Let us define

$$f(x) = \begin{cases} u^0(x) & 0 \le x \le 1, \\ -u^0(-x) & -1 \le x \le 0. \end{cases} \tag{20.8}$$

$f(x)$ is an odd 2-periodic extension of $u^0(x)$ in $(-1,1)$.

$$0 = 2\int_0^1 u^0(x)\cos(m\pi x)dx.$$

$$a_m = 2\int_0^1 u^0(x)\sin(m\pi x)dx.$$

Thus a_m is the coefficient in the Fourier sine series expansion of $u^0(x)$. Evaluation by integration on $[0,\frac{1}{2}]$ and $[\frac{1}{2},0]$

$$a_m = 2\int_0^{\frac{1}{2}} u^0(x)\sin(m\pi x)dx + 2\int_{\frac{1}{2}}^1 u^0(x)\sin(m\pi x)dx$$

using the following lemma:

$$\int z\sin(az)dz = \frac{\sin(az)}{a^2} - \frac{z\cos(az)}{a} + C.$$

First integral:

$$\begin{aligned}
2\int_0^{\frac{1}{2}} 2x\sin(m\pi x)dx &= 4\int_0^{\frac{1}{2}} x\sin(m\pi x)dx \\
&= 4\left[\frac{\sin(m\pi x)}{m^2\pi^2} - \frac{x\cos(m\pi x)}{m\pi}\right]_0^{\frac{1}{2}} \\
&= 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi}.
\end{aligned}$$

Second integral:

$$\begin{aligned}
2\int_{\frac{1}{2}}^1 2\sin(m\pi x)dx &= 4\left[\frac{-1}{m\pi}\cos(m\pi x)\right]_{\frac{1}{2}}^1 \\
&= \frac{-4}{m\pi}\cos(m\pi 1) + \frac{4}{m\pi}\cos(m\pi\frac{1}{2})
\end{aligned}$$

Third integral:

$$\begin{aligned}
2\int_{\frac{1}{2}}^1 -2x\sin(m\pi x)dx &= -4\int_{\frac{1}{2}}^1 x\sin(m\pi x)dx \\
&= -4\left[\frac{\sin(m\pi x)}{m^2\pi^2} - \frac{x\cos(m\pi x)}{m\pi}\right]_{\frac{1}{2}}^1 \\
&= -4\frac{\sin(m\pi 1)}{m^2\pi^2} + \frac{4\cos(m\pi 1)}{m\pi} \\
&+ 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi}.
\end{aligned}$$

From the first and third integral

$$4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} + 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} = 8\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2}.$$

From the first, second and third

$$-2\frac{\cos(m\pi\frac{1}{2})}{m\pi} + \frac{4}{m\pi}\cos(m\pi\frac{1}{2}) - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi} = 0$$

From the second and third

$$-\frac{4}{m\pi}\cos(m\pi 1) + \frac{4}{m\pi}\cos(m\pi 1) = 0$$

and finally

$$4\frac{\sin(m\pi)}{m^2\pi^2} = 0.$$

$$u^0(x) = \frac{8}{\pi^2}\left(\frac{\sin(\pi x)}{1^2} - \frac{\sin(3\pi x)}{3^2} + \frac{\sin(5\pi x)}{5^2}\dots\right)$$

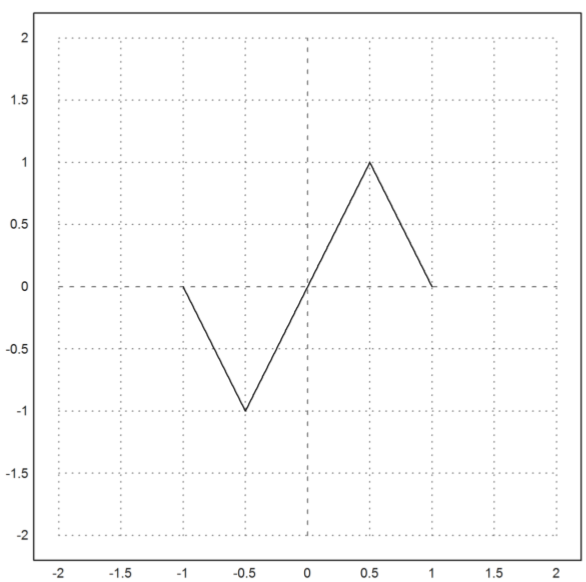
Crude approximation

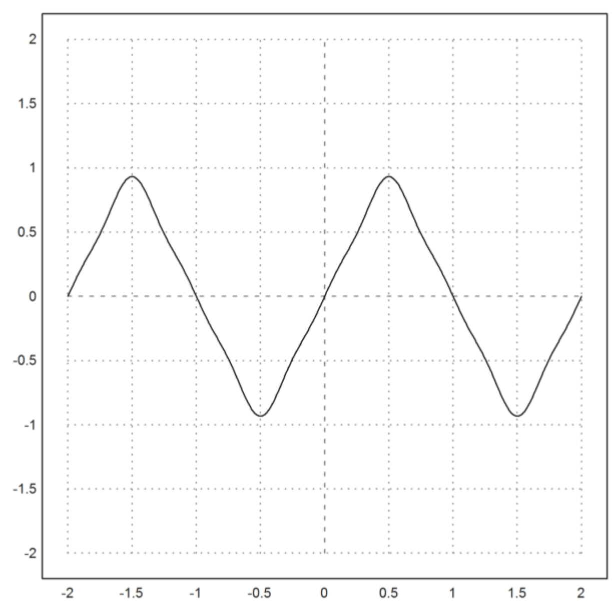
Input function $f(x)$ is is an odd 2-periodic extension of $u^0(x)$ in $[-1, 1]$, $f(x)$ is continuous, however $f'(x)$ is discontinuous at $x = \pm 0.5$. Therefore the truncated sum

$$g_m(x) = \sum_{k=1}^{m} \frac{8}{k^2\pi^2} \sin\left(\frac{k\pi}{2}\right) \sin(k\pi x)$$

can approximate $f(x)$ in $[-1, 1]$ to any degree of desired accuracy as m tends to ∞. Furthermore, $a_k \sim \dfrac{1}{k^2}$.

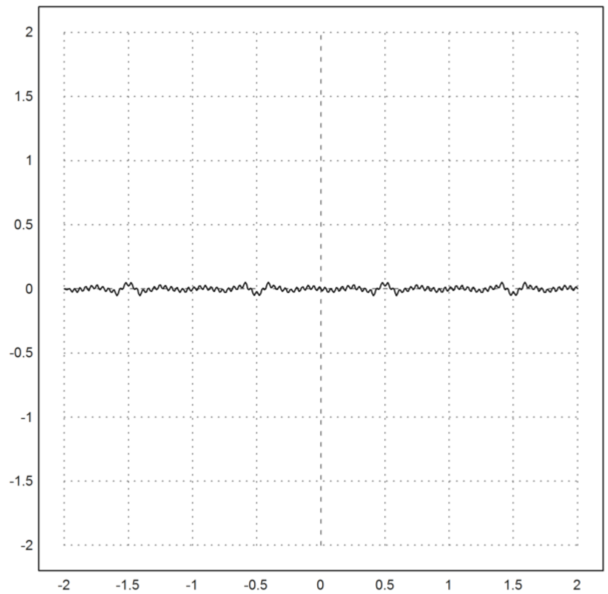
Next, we shall inspect the crude approximation generated by a Fourier sine series when truncated after 3 non-zero terms.

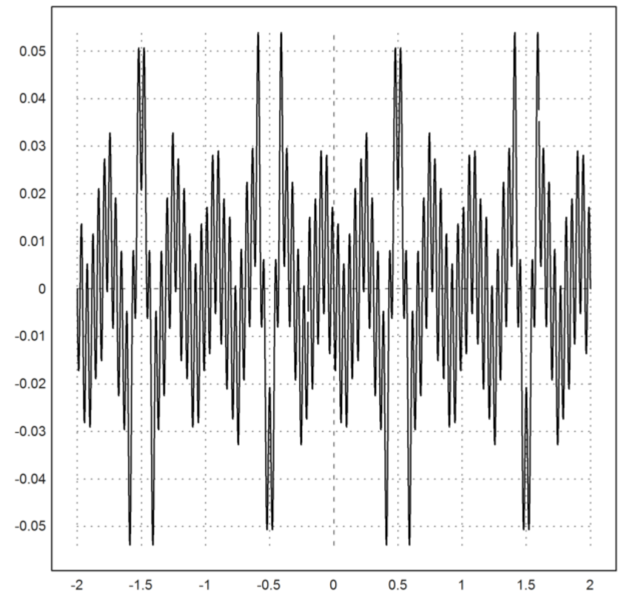




Below the diagram on the left shows the error in the original coordinate frame, the diagram on the right has a scaled frame. The ringing effect, also known as Gibbs phenomenon, is present at $x = \pm 0.5 \pm 1$.

The Fourier series is proven to be optimal in the least-squared sense, but it requires many terms to approximate sharp corners.





We shall demonstrate this fact by numerical examples.

Two kinds of errors in approximation

Write

$$e(x) = f(x) - g(x)$$

for the error of the approximation by the truncated series $g_m(x)$.

$$E_m^2 = ||e||^2 = \int_{-L}^{L} |f(x) - g(x)|^2 dx, \quad \text{L2 norm of } e(x).$$

By the orthogonality of the terms in $g_m(x)$

$$E_m^2 = ||e||^2 = ||f(x) - g_m(x)||^2 = ||f(x)||^2 - ||g_m(x)||^2$$

Note that both $||f(x)||^2$ and $||g_m(x)||^2$ can be calculated easily. Moreover

$$D_m = \max |f(x) - g_m(x)| \text{ L1 norm of } e(x).$$

We shall observe these two kinds of errors by elementary statistical methods for $m = 5, 10, 20, 40$.

Numerical analysis of 4 cases

Case 1, m=5

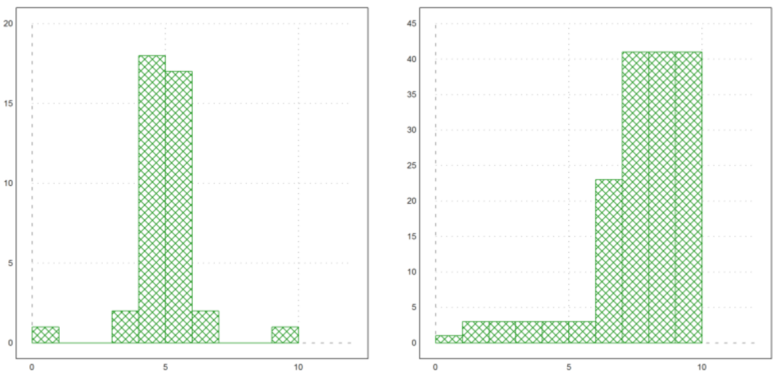
Morton-Mayers 2.1 Approximations, m= 5

x	f(x)	g(x)	error	err**2
-1.0000	0.000000E+00	0.906169E-07	-0.906169E-07	0.821142E-14
-0.9500	-0.100000E+00	-0.108839E+00	0.883943E-02	0.781355E-04
-0.9000	-0.200000E+00	-0.210040E+00	0.100397E-01	0.100796E-03
-0.8500	-0.300000E+00	-0.301963E+00	0.196269E-02	0.385216E-05
-0.8000	-0.400000E+00	-0.390785E+00	-0.921455E-02	0.849079E-04
-0.7500	-0.500000E+00	-0.486548E+00	-0.134516E-01	0.180946E-03
-0.7000	-0.600000E+00	-0.595511E+00	-0.448942E-02	0.201549E-04
-0.6500	-0.700000E+00	-0.713385E+00	0.133852E-01	0.179165E-03
-0.6000	-0.800000E+00	-0.823835E+00	0.238351E-01	0.568113E-03
-0.5500	-0.900000E+00	-0.903763E+00	0.376326E-02	0.141621E-04
-0.5000	-0.100000E+01	-0.933055E+00	-0.669445E-01	0.448157E-02
-0.4500	-0.900000E+00	-0.903763E+00	0.376332E-02	0.141626E-04
-0.4000	-0.800000E+00	-0.823835E+00	0.238352E-01	0.568116E-03
-0.3500	-0.700000E+00	-0.713385E+00	0.133853E-01	0.179166E-03
-0.3000	-0.600000E+00	-0.595511E+00	-0.448942E-02	0.201549E-04
-0.2500	-0.500000E+00	-0.486548E+00	-0.134516E-01	0.180946E-03
-0.2000	-0.400000E+00	-0.390785E+00	-0.921452E-02	0.849074E-04
-0.1500	-0.300000E+00	-0.301963E+00	0.196275E-02	0.385239E-05
-0.1000	-0.200000E+00	-0.210040E+00	0.100398E-01	0.100797E-03
-0.0500	-0.100000E+00	-0.108839E+00	0.883949E-02	0.781365E-04
0.0000	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.0500	0.100000E+00	0.108839E+00	-0.883949E-02	0.781365E-04
0.1000	0.200000E+00	0.210040E+00	-0.100398E-01	0.100797E-03
0.1500	0.300000E+00	0.301963E+00	-0.196275E-02	0.385239E-05
0.2000	0.400000E+00	0.390785E+00	0.921452E-02	0.849074E-04
0.2500	0.500000E+00	0.486548E+00	0.134516E-01	0.180946E-03
0.3000	0.600000E+00	0.595511E+00	0.448942E-02	0.201549E-04
0.3500	0.700000E+00	0.713385E+00	-0.133853E-01	0.179166E-03
0.4000	0.800000E+00	0.823835E+00	-0.238352E-01	0.568116E-03

```
 0.4500    0.900000E+00    0.903763E+00   -0.376332E-02    0.141626E-04
 0.5000    0.100000E+01    0.933055E+00    0.669445E-01    0.448157E-02
 0.5500    0.900000E+00    0.903763E+00   -0.376326E-02    0.141621E-04
 0.6000    0.800000E+00    0.823835E+00   -0.238351E-01    0.568113E-03
 0.6500    0.700000E+00    0.713385E+00   -0.133852E-01    0.179165E-03
 0.7000    0.600000E+00    0.595511E+00    0.448942E-02    0.201549E-04
 0.7500    0.500000E+00    0.486548E+00    0.134516E-01    0.180946E-03
 0.8000    0.400000E+00    0.390785E+00    0.921455E-02    0.849079E-04
 0.8500    0.300000E+00    0.301963E+00   -0.196269E-02    0.385216E-05
 0.9000    0.200000E+00    0.210040E+00   -0.100397E-01    0.100796E-03
 0.9500    0.100000E+00    0.108839E+00   -0.883943E-02    0.781355E-04
 1.0000    0.000000E+00   -0.906169E-07    0.906169E-07    0.821142E-14
Histogram of errors:
-0.540912E-01        1
-0.405684E-01        0
-0.270456E-01        0
-0.135228E-01        2
0.000000E+00        18
0.135228E-01        17
0.270456E-01         2
0.405684E-01         0
0.540912E-01         0
0.676140E-01         1
Cumulative distr. of abs. errors:
0.100000E-07         1
0.100000E-06         3
0.100000E-05         3
0.100000E-04         3
0.100000E-03         3
0.100000E-02         3
0.100000E-01        23
0.100000E+00        41
0.100000E+01        41
0.100000E+02        41
Results:
Truncated at  m=    5
dx=  0.500000E-01
abs max error=  0.669445E-01
```

```
integral of f**2=  0.666667E+00
integral of g**2=  0.666185E+00
int. of error**2=  0.481248E-03
same, checked  =  0.481174E-03
least non-zero =  0.324228E-01
```

Histogram of errors and cumulative distribution of absolute errors:



Case 2, m=10

Morton-Mayers 2.1 Approximations, m= 10

x	f(x)	g(x)	error	err**2
-1.0000	0.000000E+00	0.693778E-07	-0.693778E-07	0.481327E-14
-0.9500	-0.100000E+00	-0.103984E+00	0.398402E-02	0.158724E-04
-0.9000	-0.200000E+00	-0.199749E+00	-0.250906E-03	0.629537E-07
-0.8500	-0.300000E+00	-0.295634E+00	-0.436589E-02	0.190610E-04
-0.8000	-0.400000E+00	-0.400636E+00	0.636041E-03	0.404548E-06
-0.7500	-0.500000E+00	-0.505322E+00	0.532156E-02	0.283190E-04
-0.7000	-0.600000E+00	-0.598495E+00	-0.150537E-02	0.226615E-05
-0.6500	-0.700000E+00	-0.692503E+00	-0.749648E-02	0.561972E-04
-0.6000	-0.800000E+00	-0.804595E+00	0.459456E-02	0.211100E-04
-0.5500	-0.900000E+00	-0.912839E+00	0.128387E-01	0.164833E-03
-0.5000	-0.100000E+01	-0.959605E+00	-0.403953E-01	0.163178E-02
-0.4500	-0.900000E+00	-0.912839E+00	0.128388E-01	0.164836E-03
-0.4000	-0.800000E+00	-0.804595E+00	0.459468E-02	0.211111E-04
-0.3500	-0.700000E+00	-0.692504E+00	-0.749636E-02	0.561954E-04
-0.3000	-0.600000E+00	-0.598495E+00	-0.150537E-02	0.226615E-05
-0.2500	-0.500000E+00	-0.505322E+00	0.532156E-02	0.283190E-04

-0.2000	-0.400000E+00	-0.400636E+00	0.636071E-03	0.404586E-06
-0.1500	-0.300000E+00	-0.295634E+00	-0.436580E-02	0.190602E-04
-0.1000	-0.200000E+00	-0.199749E+00	-0.250816E-03	0.629088E-07
-0.0500	-0.100000E+00	-0.103984E+00	0.398407E-02	0.158728E-04
0.0000	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.0500	0.100000E+00	0.103984E+00	-0.398407E-02	0.158728E-04
0.1000	0.200000E+00	0.199749E+00	0.250816E-03	0.629088E-07
0.1500	0.300000E+00	0.295634E+00	0.436580E-02	0.190602E-04
0.2000	0.400000E+00	0.400636E+00	-0.636071E-03	0.404586E-06
0.2500	0.500000E+00	0.505322E+00	-0.532156E-02	0.283190E-04
0.3000	0.600000E+00	0.598495E+00	0.150537E-02	0.226615E-05
0.3500	0.700000E+00	0.692504E+00	0.749636E-02	0.561954E-04
0.4000	0.800000E+00	0.804595E+00	-0.459468E-02	0.211111E-04
0.4500	0.900000E+00	0.912839E+00	-0.128388E-01	0.164836E-03
0.5000	0.100000E+01	0.959605E+00	0.403953E-01	0.163178E-02
0.5500	0.900000E+00	0.912839E+00	-0.128387E-01	0.164833E-03
0.6000	0.800000E+00	0.804595E+00	-0.459456E-02	0.211100E-04
0.6500	0.700000E+00	0.692503E+00	0.749648E-02	0.561972E-04
0.7000	0.600000E+00	0.598495E+00	0.150537E-02	0.226615E-05
0.7500	0.500000E+00	0.505322E+00	-0.532156E-02	0.283190E-04
0.8000	0.400000E+00	0.400636E+00	-0.636041E-03	0.404548E-06
0.8500	0.300000E+00	0.295634E+00	0.436589E-02	0.190610E-04
0.9000	0.200000E+00	0.199749E+00	0.250906E-03	0.629537E-07
0.9500	0.100000E+00	0.103984E+00	-0.398402E-02	0.158724E-04
1.0000	0.000000E+00	-0.693778E-07	0.693778E-07	0.481327E-14

Histogram of errors:

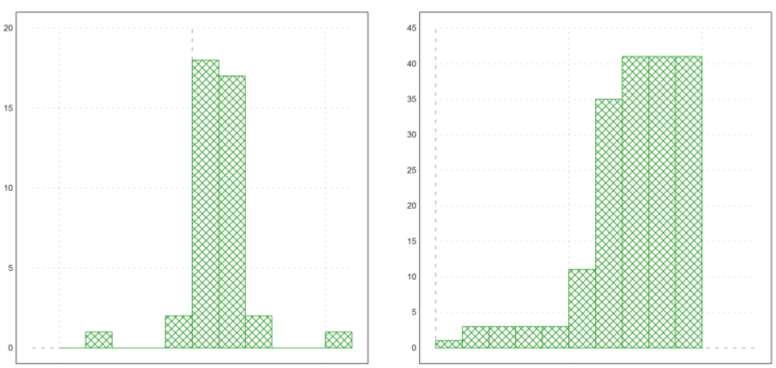
-0.326394E-01	1
-0.244796E-01	0
-0.163197E-01	0
-0.815985E-02	2
0.000000E+00	18
0.815985E-02	17
0.163197E-01	2
0.244796E-01	0
0.326394E-01	0
0.407993E-01	1

Cumulative distr. of abs. errors:

0.100000E-07	1

```
0.100000E-06         3
0.100000E-05         3
0.100000E-04         3
0.100000E-03         3
0.100000E-02        11
0.100000E-01        35
0.100000E+00        41
0.100000E+01        41
0.100000E+02        41
Results:
Truncated at  m=   10
dx=  0.500000E-01
abs max error=  0.403953E-01
integral of f**2=  0.666667E+00
integral of g**2=  0.666559E+00
int. of error**2=  0.107408E-03
same, checked  =  0.107389E-03
least non-zero =  0.100070E-01
```

Histogram of errors and cumulative distribution of absolute errors:



Morton-Mayers 2.1 Approximations, m= 20

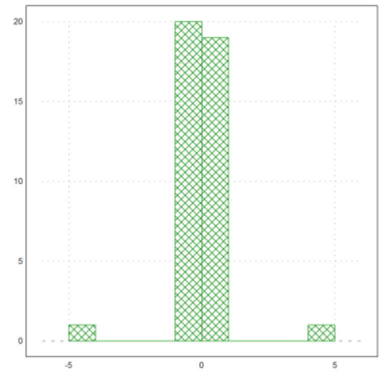
x	f(x)	g(x)	error	err**2
-1.0000	0.000000E+00	0.622382E-07	-0.622382E-07	0.387359E-14
-0.9500	-0.100000E+00	-0.100016E+00	0.157878E-04	0.249254E-09
-0.9000	-0.200000E+00	-0.199966E+00	-0.337958E-04	0.114216E-08
-0.8500	-0.300000E+00	-0.300056E+00	0.560880E-04	0.314586E-08
-0.8000	-0.400000E+00	-0.399912E+00	-0.876486E-04	0.768228E-08

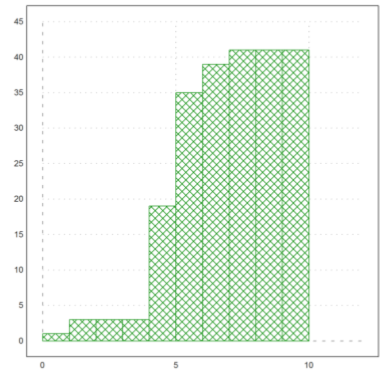
-0.7500	-0.500000E+00	-0.500136E+00	0.136137E-03	0.185333E-07
-0.7000	-0.600000E+00	-0.599779E+00	-0.220656E-03	0.486892E-07
-0.6500	-0.700000E+00	-0.700390E+00	0.389993E-03	0.152095E-06
-0.6000	-0.800000E+00	-0.799188E+00	-0.811577E-03	0.658657E-06
-0.5500	-0.900000E+00	-0.902349E+00	0.234914E-02	0.551845E-05
-0.5000	-0.100000E+01	-0.979752E+00	-0.202475E-01	0.409962E-03
-0.4500	-0.900000E+00	-0.902349E+00	0.234926E-02	0.551901E-05
-0.4000	-0.800000E+00	-0.799188E+00	-0.811517E-03	0.658560E-06
-0.3500	-0.700000E+00	-0.700390E+00	0.390112E-03	0.152188E-06
-0.3000	-0.600000E+00	-0.599779E+00	-0.220716E-03	0.487156E-07
-0.2500	-0.500000E+00	-0.500136E+00	0.136137E-03	0.185333E-07
-0.2000	-0.400000E+00	-0.399912E+00	-0.876188E-04	0.767706E-08
-0.1500	-0.300000E+00	-0.300056E+00	0.561476E-04	0.315255E-08
-0.1000	-0.200000E+00	-0.199966E+00	-0.337064E-04	0.113612E-08
-0.0500	-0.100000E+00	-0.100016E+00	0.158325E-04	0.250668E-09
0.0000	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.0500	0.100000E+00	0.100016E+00	-0.158325E-04	0.250668E-09
0.1000	0.200000E+00	0.199966E+00	0.337064E-04	0.113612E-08
0.1500	0.300000E+00	0.300056E+00	-0.561476E-04	0.315255E-08
0.2000	0.400000E+00	0.399912E+00	0.876188E-04	0.767706E-08
0.2500	0.500000E+00	0.500136E+00	-0.136137E-03	0.185333E-07
0.3000	0.600000E+00	0.599779E+00	0.220716E-03	0.487156E-07
0.3500	0.700000E+00	0.700390E+00	-0.390112E-03	0.152188E-06
0.4000	0.800000E+00	0.799188E+00	0.811517E-03	0.658560E-06
0.4500	0.900000E+00	0.902349E+00	-0.234926E-02	0.551901E-05
0.5000	0.100000E+01	0.979752E+00	0.202475E-01	0.409962E-03
0.5500	0.900000E+00	0.902349E+00	-0.234914E-02	0.551845E-05
0.6000	0.800000E+00	0.799188E+00	0.811577E-03	0.658657E-06
0.6500	0.700000E+00	0.700390E+00	-0.389993E-03	0.152095E-06
0.7000	0.600000E+00	0.599779E+00	0.220656E-03	0.486892E-07
0.7500	0.500000E+00	0.500136E+00	-0.136137E-03	0.185333E-07
0.8000	0.400000E+00	0.399912E+00	0.876486E-04	0.768228E-08
0.8500	0.300000E+00	0.300056E+00	-0.560880E-04	0.314586E-08
0.9000	0.200000E+00	0.199966E+00	0.337958E-04	0.114216E-08
0.9500	0.100000E+00	0.100016E+00	-0.157878E-04	0.249254E-09
1.0000	0.000000E+00	-0.622382E-07	0.622382E-07	0.387359E-14

```
Histogram of errors:
-0.163600E-01         1
```

```
-0.122700E-01         0
-0.818000E-02         0
-0.409000E-02         0
0.000000E+00        20
0.409000E-02        19
0.818000E-02         0
0.122700E-01         0
0.163600E-01         0
0.204500E-01         1
Cumulative distr. of abs. errors:
0.100000E-07         1
0.100000E-06         3
0.100000E-05         3
0.100000E-04         3
0.100000E-03        19
0.100000E-02        35
0.100000E-01        39
0.100000E+00        41
0.100000E+01        41
0.100000E+02        41
Results:
Truncated at  m=   20
dx=  0.500000E-01
abs max error=  0.202475E-01
integral of f**2=  0.666667E+00
integral of g**2=  0.666653E+00
int. of error**2=  0.137091E-04
same, checked  =  0.136251E-04
least non-zero =  0.224534E-02
```

Histogram of errors and cumulative distribution of absolute errors:





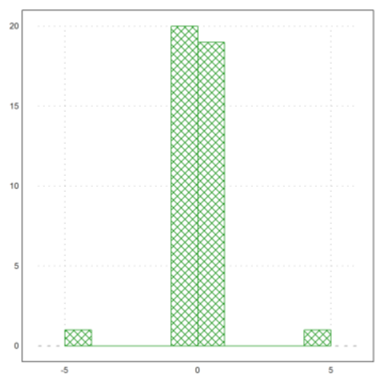
Morton-Mayers 2.1 Approximations, m= 40

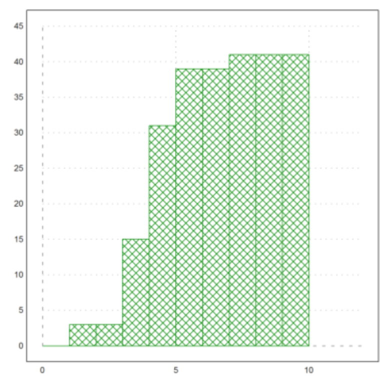
x	f(x)	g(x)	error	err**2
-1.0000	0.000000E+00	0.637402E-07	-0.637402E-07	0.406281E-14
-0.9500	-0.100000E+00	-0.999980E-01	-0.205636E-05	0.422862E-11
-0.9000	-0.200000E+00	-0.199996E+00	-0.442564E-05	0.195863E-10
-0.8500	-0.300000E+00	-0.299993E+00	-0.721216E-05	0.520153E-10
-0.8000	-0.400000E+00	-0.399989E+00	-0.113547E-04	0.128929E-09
-0.7500	-0.500000E+00	-0.499982E+00	-0.177026E-04	0.313381E-09
-0.7000	-0.600000E+00	-0.599971E+00	-0.290871E-04	0.846057E-09
-0.6500	-0.700000E+00	-0.699947E+00	-0.530481E-04	0.281410E-08
-0.6000	-0.800000E+00	-0.799882E+00	-0.117838E-03	0.138859E-07
-0.5500	-0.900000E+00	-0.899588E+00	-0.412464E-03	0.170127E-06
-0.5000	-0.100000E+01	-0.989870E+00	-0.101300E-01	0.102618E-03
-0.4500	-0.900000E+00	-0.899588E+00	-0.412345E-03	0.170028E-06
-0.4000	-0.800000E+00	-0.799882E+00	-0.117779E-03	0.138718E-07
-0.3500	-0.700000E+00	-0.699947E+00	-0.529289E-04	0.280147E-08
-0.3000	-0.600000E+00	-0.599971E+00	-0.291467E-04	0.849528E-09
-0.2500	-0.500000E+00	-0.499982E+00	-0.177026E-04	0.313381E-09
-0.2000	-0.400000E+00	-0.399989E+00	-0.113249E-04	0.128253E-09
-0.1500	-0.300000E+00	-0.299993E+00	-0.718236E-05	0.515863E-10
-0.1000	-0.200000E+00	-0.199996E+00	-0.432134E-05	0.186740E-10
-0.0500	-0.100000E+00	-0.999980E-01	-0.204146E-05	0.416756E-11
0.0000	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.0500	0.100000E+00	0.999980E-01	0.204146E-05	0.416756E-11
0.1000	0.200000E+00	0.199996E+00	0.432134E-05	0.186740E-10
0.1500	0.300000E+00	0.299993E+00	0.718236E-05	0.515863E-10
0.2000	0.400000E+00	0.399989E+00	0.113249E-04	0.128253E-09

```
 0.2500     0.500000E+00    0.499982E+00    0.177026E-04    0.313381E-09
 0.3000     0.600000E+00    0.599971E+00    0.291467E-04    0.849528E-09
 0.3500     0.700000E+00    0.699947E+00    0.529289E-04    0.280147E-08
 0.4000     0.800000E+00    0.799882E+00    0.117779E-03    0.138718E-07
 0.4500     0.900000E+00    0.899588E+00    0.412345E-03    0.170028E-06
 0.5000     0.100000E+01    0.989870E+00    0.101300E-01    0.102618E-03
 0.5500     0.900000E+00    0.899588E+00    0.412464E-03    0.170127E-06
 0.6000     0.800000E+00    0.799882E+00    0.117838E-03    0.138859E-07
 0.6500     0.700000E+00    0.699947E+00    0.530481E-04    0.281410E-08
 0.7000     0.600000E+00    0.599971E+00    0.290871E-04    0.846057E-09
 0.7500     0.500000E+00    0.499982E+00    0.177026E-04    0.313381E-09
 0.8000     0.400000E+00    0.399989E+00    0.113547E-04    0.128929E-09
 0.8500     0.300000E+00    0.299993E+00    0.721216E-05    0.520153E-10
 0.9000     0.200000E+00    0.199996E+00    0.442564E-05    0.195863E-10
 0.9500     0.100000E+00    0.999980E-01    0.205636E-05    0.422862E-11
 1.0000     0.000000E+00   -0.637402E-07    0.637402E-07    0.406281E-14
Histogram of errors:
-0.818508E-02        1
-0.613881E-02        0
-0.409254E-02        0
-0.204627E-02        0
0.000000E+00        20
0.204627E-02        19
0.409254E-02         0
0.613881E-02         0
0.818508E-02         0
0.102313E-01         1
Cumulative distr. of abs. errors:
0.100000E-07         1
0.100000E-06         3
0.100000E-05         3
0.100000E-04        15
0.100000E-03        31
0.100000E-02        39
0.100000E-01        39
0.100000E+00        41
0.100000E+01        41
0.100000E+02        41
```

```
Results:
Truncated at  m=   40
dx=  0.500000E-01
abs max error=  0.101300E-01
integral of f**2=  0.666667E+00
integral of g**2=  0.666665E+00
int. of error**2=  0.178814E-05
same, checked  =  0.171895E-05
least non-zero =  0.532919E-03
```

Histogram of errors and cumulative distribution of absolute errors:





Summary:

m	E_m^2	D_m
5	0.482448E-03	0.669445E-01
10	0.107408E-03	0.403953E-01
20	0.137091E-04	0.202475E-01
40	0.178814E-05	0.101300E-01

Approximation within 0.001

Write $\eta = 0.001$ and

$$t_m = \sum_{k=m}^{\infty} a_k \sin(k\pi x),$$

our objective is conceptually different now, we want to determine m, so that

$$\eta > D_m = \max |t_m(x)|.$$

$$|t_m(x)| = |\sum_{k=m}^{\infty} a_k \sin(k\pi x)| \leq \sum_{k=m}^{\infty} |a_k| |\sin(k\pi x)|$$

$$|\sin(k\pi x)| \leq 1$$

$$a_k = \frac{8}{k^2\pi^2} \sin\left(\frac{k\pi}{2}\right) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \pm 1 & \text{if } k \text{ is odd} \end{cases}$$

Therefore it suffices to consider only odd indexes. Let $k = 2p+1$, where p runs through $0, 1, \ldots$. So our task is simplified : find an upper bound for

$$S_0 = \sum_{p=p_0}^{\infty} \frac{1}{(2p+1)^2}.$$

Next, we show

$$(ii),\ \int_{2p}^{2p+2} \frac{dx}{x^2} > \frac{2}{(2p+1)^2}.$$

$$\int_{2p}^{2p+2} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{2p}^{2p+2} = -\frac{1}{2p+2} + \frac{1}{2p}$$

$$\frac{1}{2p} - \frac{1}{2p+2} = \frac{2}{2p(2p+2)} = \frac{2}{4p^2+4p} > \frac{2}{4p^2+4p+1} = \frac{2}{(2p+1)^2}.$$

Further

$$\int_{2p+2}^{2p+4} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{2p+2}^{2p+4} = -\frac{1}{2p+4} + \frac{1}{2p+2} = \frac{-2p-2+2p+4}{(2p+2)(2p+4)}$$

$$\int_{2p+2}^{2p+4} \frac{dx}{x^2} = \frac{2}{(2p+2)(2p+4)} = \frac{2}{(2(p+1))(2(p+1)+2)}$$

$$(2(p+1)(2(p+1)+2) = 4(p+1)^2 + 4(p+1) < 4(p+1)^2 + 4(p+1) + 1$$

$$4(p+1)^2 + 4(p+1) + 1 = 4p^2 + 12p + 9 = (2p+3)^2$$

$$\int_{2p+2}^{2p+4} \frac{dx}{x^2} > \frac{2}{(2p+3)^2}.$$

Following this pattern

$$\int_{2p_0}^{2p_0+2} \frac{dx}{x^2} + \int_{2p_0+2}^{2p_0+4} \frac{dx}{x^2} + \ldots > \frac{2}{(2p_0+1)^2} + \frac{2}{(2p_0+3)^2} + \ldots$$

By the additive property of the integral

$$\frac{1}{2}\int_{2p_0}^{\infty}\frac{dx}{x^2} > \sum_{p=p_0}^{\infty}\frac{1}{(2p+1)^2}.$$

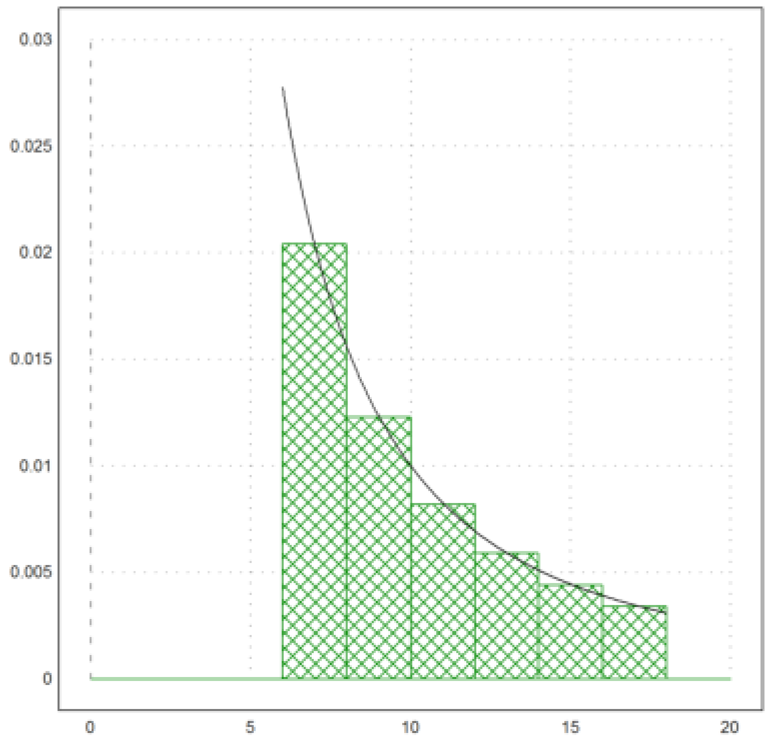
Upon evaluating the improper integral on the left hand side we obtain

$$\frac{1}{4p_0} > \sum_{p=p_0}^{\infty}\frac{1}{(2p+1)^2}.$$

Below is the calculation of $|f(x) - g_m(x)|$. If we stop at $m = 401$ the error is still bigger than 0.001. In fact, we need $m = 405$ for the error to be less than 0.001.

```
 Morton-Mayers 2.1 L1 Approximations at x=pi/2
 Bound =  1.000000E-03
 m stop       g(pi/2)         L1 error
   1      0.810569E+00     0.189431E+00
 101      0.996027E+00     0.397331E-02
 201      0.997994E+00     0.200641E-02
 301      0.998658E+00     0.134206E-02
 401      0.998992E+00     0.100839E-02
 501      0.999192E+00     0.807524E-03
 601      0.999327E+00     0.673413E-03
 701      0.999422E+00     0.577569E-03
 801      0.999494E+00     0.505567E-03
 901      0.999551E+00     0.449419E-03
1001      0.999596E+00     0.404477E-03
```

Further, let us consider the rectangles in the positive quadrant of the coordinate system. Let the $k-th$ rectangle represent $|a_k|$, then $\sum_{k=m}^{\infty}|a_k|$ can be approximated by a definite integral of an elementary function, $f(x) = 1/x^2$, which gives a good, closed analytical formula for the upper bound of the infinite sum. (Only six rectangles are displayed below:)



Codes

Euler Notebook sample, abbreviated

```
>function f(x):= sin(1*pi*x)/1^2-sin(3*pi*x)/3^2+sin(5*pi*x)/5^2
>function f(x):= f(x)-sin(7*pi*x)/7^2+sin(9*pi*x)/9^2-sin(11*pi*x)/11^2
>Function f(x):= f(x)+sin(13*pi*x)/13^2-sin(15*pi*x)/15^2;
...
>function f(x) := ( f(x)+sin(43*pi*x)/43^2-sin(45*pi*x)/45)*8/(pi*pi);
>plot2d("f(x)"):
```

Omnibus Fortran 90/95 program

```
    Program main
!==============================================
! Typical approximation for Exercise 1 in MM.
!
```

```
! Objectives:
! Calculate g(x) the truncated F-series for f(x)
! Display results and errors of approximation, |f(x)-g(x)|
! Calculate squared error, integral(|f(x)-g(x)|^2)
! Find max(|f(x)-g(x)|), demonstrate Gibbs - phenom.
! Integration of a function using Simpson rule
!
! check, Dec 31, 2019
! integration by simple simpson-method, single prcision
! paper and pencil method, minimalist approach
!
!==============================================
!123456789==========================================
implicit none                                    ! external functions
real, external :: f1, f2, g1, g2, h2, g3
integer, parameter ::  np=20, m=40              ! this needs to be checked
real, parameter :: pi=3.1415927                  ! parameters of integration
real ::  a, b, integral, dx, xdummy
integer:: n, i, ierror                           ! miscellanious arrays
real, dimension (-np:np) :: y, s, xp, e, ee      ! error calculation
real :: total, emax
integer, parameter :: nbin = 11                  ! histogram
real, dimension(nbin)   :: bin
real t0k,t0kp1,t1k                               ! Richardson extrapolation
integer  k, kp1
!
real, dimension (0:10) :: res                    ! results

! res(0) - m, truncation
! res(1) - dx
! res(2) - emax
! res(3) - integral(f*f)
! res(4) - integral(g*g)
! res(5) - error squared, as difference
! res(6) - error squared, direct integration
!
!123456789==========================================
!
```

```
open(unit=10,file="MM21.dat",status="old",action="write",iostat=ierror)
if(ierror/=0) then
print*, "failed to open MM21.dat"
stop
else
print*, "  ***   opened MM21.dat"
end if
!
res(0)=float(m)
a = -1.0
b =  1.0
dx=(b-a)/(np+np)
res(1)=dx
!==============================================================
! PART I. Check approximation:
!===================================================
emax=0.0
write(10,1000) m
1000      format('   Morton-Mayers 2.1 Approximations, m=',i5)
write(10,1001)
1001      format('   x          f(x)             g(x)             error
do i= -np,np, 1
xp(i)=i*dx
y(i)=f1(xp(i))
s(i)=g1(xp(i))
e(i)=f1(xp(i))-g1(xp(i))
ee(i)=e(i)**2
emax=max(emax,e(i))
write(10,1002) xp(i),y(i),s(i), e(i), ee(i)
1002      format(f9.4,3x,E14.6,3x,E14.6,3x,E14.6,3x,E14.6)
total = total + ee(i)
end do
write(*,105)   emax
res(2)=emax
!==============================================================
! PART II.
! calculate integral(f**2) , integral(g**2) and error**2
!==================================================
```

```
n = 2
write(*,100)
do i=1,8
call simpson(f2,a,b,integral,n)
write (*,101) n, integral
n = n*2
end do
res(3)=integral
!
!  calculate integral(g**2) by Parseval_Theorem
!
xdummy=0.0
total=g2(xdummy)
write(*,130)   total
!
res(4)=total
res(5)=integral-total
write(*,140)   res(5)
! ===============================================================
!  PART III.
!  Romberg approx of error**2, double check
! Test of numerical integration of
! h(x)=|f(x)-g(x)|**2
!
!  Richardson extrapolation
!
t0k=0.0
t0kp1=0.0
!
k=8
call trapzd(h2,-1.0,1.0,t0k,k)
write(*,141)   t0k
kp1=k+1
t0kp1=0.0
call trapzd(h2,-1.0, 1.0,t0kp1,kp1)
write(*,141)   t0kp1
!
t1k=(4.0*t0kp1-t0k)/3.0
```

```
write(*,141)   t1k

!
res(6)=t1k
res(7)=g3(m)
! =============================================================
!  PART IV.
!  Histograms of error
! =============================================================
!
write(10,1997)
!
!       first
!
bin(6)=0.0
bin(7)=((emax*1.01)/5.0)
bin(8)=((emax*1.01)/5.0)*2.0
bin(9)=((emax*1.01)/5.0)*3.0
bin(10)=((emax*1.01)/5.0)*4.0
bin(11)=((emax*1.01)/5.0)*5.0
bin(5)=-bin(7)
bin(4)=-bin(8)
bin(3)=-bin(9)
bin(2)=-bin(10)
bin(1)=-bin(11)

call histo( e, bin, np+np+1, 11 )
!
!       second
!
bin(1)=1.0e-09
bin(2)=1.0e-08
bin(3)=1.0e-07
bin(4)=1.0e-06
bin(5)=1.0e-05
bin(6)=1.0e-04
bin(7)=1.0e-03
bin(8)=1.0e-02
```

```
bin(9)=1.0e-01
bin(10)=1.0e-00
bin(11)=1.0e+01

!
write(10,1998)
call histo2( abs(e), bin, np+np+1, 11 )
!
1997      format(' Histogram of errors:')
1998      format(' Cumulative distr. of abs. errors:')
!
!
write(10,1999)
1999       format(' Results:')
!! res(0) - m, truncation
write(10,2000) m
2000       format(' Truncated at  m=',i5)
! res(1) - dx
write(10,2002) res(1)
2002       format('              dx=',e14.6)
! res(2) - emax
write(10,2003) res(2)
2003       format('   abs max error=',e14.6)
! res(3) - integral(f*f)
write(10,2004) res(3)
2004       format('integral of f**2=',e14.6)
! res(4) - integral(g*g)
write(10,2005) res(4)
2005       format('integral of g**2=',e14.6)
! res(5) - error squared, as difference
write(10,2006) res(5)
2006       format('int. of error**2=',e14.6)
write(10,2007) res(6)
2007       format(' same, checked  =',e14.6)
write(10,2008) res(7)
2008       format(' least non-zero =',e14.6)
!
close(unit=10)
```

```
!
print*, "  ***   closed MM21.dat"


100   format('     nint   Simpson f2')
101   format(i9,1pe15.6)
!
105   format('     max |f(x)-g(x)| =', e14.6)
! 110   format('     nint   Simpson g2')
130   format('     Parseval  g2=', E14.6)
140   format('     Par      err=', E14.6)
141   format('     Romberg  err=', E14.6)
!
end program
!==============================================

function f1(x)
!
! input function
!
implicit none
real f1, x
!
if( x>=-1.0.and.x<=-0.5 ) then
f1= -2.0*x-2.0
endif
!
if(x>=-0.5.and.x<=0.5) then
f1= 2.0*x
endif
!
if(x>= 0.5.and.x<=1.0) then
f1 = -2.0*x+2.0
endif
!
return
end
!
```

```
!==============================================
function f2(x)
!
! input function squared
!
implicit none
real f1, f2, x
external f1
!
f2=f1(x)**2
!
return
end
!==============================================
!
function g1(x)                ! truncated Fourier series
implicit none
real g1, x, accu
integer k,l
real, parameter :: pi=3.1415927
integer, parameter :: m=40       ! this needs to be checked
accu=0.0
k=1
do while ( 2*k-1 <= m)
l=2*k-1
accu=accu-(-1)**(k)* sin(l*pi*x)/(l**2)
k=k+1
end do
g1=(8.0/(pi*pi))*accu
return
end function g1
!====================================================================
function g2(x)
implicit none
real g2, accu, a, x
integer k,l
real, parameter :: pi=3.1415927
integer, parameter :: m=40        ! this needs to be checked
```

```
accu=0.0
k=1
do while ( 2*k-1 <= m)
l=2*k-1
a=1.0/(l**2)
accu=accu+a*a
k=k+1
end do
g2=(8.0/(pi*pi))*(8.0/(pi*pi))*accu
return
end function g2
!
!===================================================================
!
function g3(m)
!
! least non-zero term of sine series, m >1
!
implicit none
real g3, accu, a
integer k,l,m
real, parameter :: pi=3.1415927
accu=0.0
k=1
do while ( 2*k-1 <= m)
l=2*k-1
a=1.0/(l**2)
accu=min(accu,a)
k=k+1
end do
g3=(8.0/(pi*pi))*a
return
end function g3
!
!===================================================================
!
function h2(x)
implicit none
```

```
real  x, h2
real, external :: f1, g1
!
h2=(f1(x)-g1(x))**2
!
return
end function h2
!======================================================================
Subroutine simpson(f,a,b,integral,n)
!==========================================================
! Modified to single precision
! Integration of f(x) on [a,b]
! Method: Simpson rule for n intervals
! written by: Alex Godunov (October 2009)
! checked by: Dr Melvin No, Privatdozent (July 2019)
!----------------------------------------------------------
! IN:
! f   - Function to integrate (supplied by a user)
! a   - Lower limit of integration
! b   - Upper limit of integration
! n   - number of intervals
! OUT:
! integral - Result of integration
!==========================================================
implicit none
real f, a, b, integral,s
real h, x
integer nint
integer n, i

! if n is odd we add +1 to make it even
if((n/2)*2.ne.n) n=n+1

! loop over n (number of intervals)
s = 0.0
h = (b-a)/float(n)
do i=2, n-2, 2
x   = a+float(i)*h
```

```
s = s + 2.0*f(x) + 4.0*f(x+h)
end do
integral = (s + f(a) + f(b) + 4.0*f(a+h))*h/3.0
return
end subroutine simpson
!
! subroutine to print histogram
!
!==============================================================================
!
subroutine histo( obs, bin, m, n  )
!
!   symm. histogram
!
real, dimension(m) :: obs
real, dimension(n)   :: bin
integer :: i, k

!

do i = 2,size(bin)
k=count( obs <= bin(i) )- count( obs <= bin(i-1) )
write( *, '(e14.6,i10)' ) bin(i), k                    ! write on panel
write( 10, '(e14.6,i10)' ) bin(i), k                   ! write on record
enddo
end subroutine histo
!
!==============================================================================
!
subroutine histo2( obs, bin, m, n  )
!
!   commulative distribution
!
real, dimension(m) :: obs
real, dimension(n)   :: bin
integer :: i, k

!
```

```
do i = 2,size(bin)
k=count( obs <= bin(i) )
write( *, '(e14.6,i10)' ) bin(i), k                    ! write on panel
write( 10, '(e14.6,i10)' ) bin(i), k                   ! write on record
enddo
end subroutine histo2
!====================================================================
subroutine trapzd(func,a,b,accu0,n)
!
! straight calculation
! a,b interval is partitioned into 2**n parts (del)
! func - function to be integrated, quadrature
! a,b  - interval b>a
! accu - accumulator
! n    - parameter
!
implicit none
real a,b,x,del, accu0
real*8 accu1
integer j,n
real, external :: func
! n>1
del=(b-a)/2**n
accu1=dble(0.5*(func(a)+func(b)))
x=a
do j=1,2**n-1
x=x+del
accu1=accu1+dble(func(x))
end do
accu1=accu1*del
accu0=sngl(accu1)
!
return
end subroutine trapzd
!
!====================================================================
!
```

```
program xmpl_03
! calculates L1 error term Assignment 41
! Morton-Mayers 2.1
! direct calculation and upper bound
implicit none
integer::  i, ierror
real one, g4, s1, s2, s3, x
external g4
!
open(unit=10,file="MM21b.dat",status="old",action="write",iostat=ierror)
if(ierror/=0) then
print*, "failed to open MM21b.dat"
stop
else
print*, "  ***   opened MM21b.dat"
end if
!
one=1.0
x=0.5
s3=1.0e-3
write(10,1000)
1000       format('   Morton-Mayers 2.1 L1 Approximations at x=pi/2')
write(10,1001)  s3
1001       format('   Bound =',4ES14.6 )
write(10,1002)
1002       format('    m stop       g(pi/2)        L1 error')
do i=1,1001,100
s1=g4(x,i)
s2=one-s1
write(10,1003) i , s1, s2
1003       format(i8,3x,E14.6,3x,E14.6)
end do
!
end program
!===========================================
!                              ! function g1, modified
function g4(x,m)               ! truncated Fourier series
implicit none
```

```
real g4, x, accu
integer k,l
real, parameter :: pi=3.1415927
integer m
accu=0.0
k=1
do while ( 2*k-1 <= m)
l=2*k-1
accu=accu-(-1)**(k)* sin(l*pi*x)/(l**2)
k=k+1
end do
g4=(8.0/(pi*pi))*accu
return
end function g4
!=============================================
```