Pinter Consulting
Work in Progress
New Series Nos. 10.

J K Pinter, Dr.Tech.

October 15, 2015

Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

Introduction

Pinter Consulting of Calgary, Alberta practices Mathematics, promotes clear thinking and offers Consultations, Tutorials and Seminars in Mathematics.

Contents

Chapter 11

Proceedings

11.1 Summary of Current Report

- **Private study for professional development:**
- Records of activities at Pinter Consulting
- Collection of problems with our own solutions .
- Continuous improvement, corrections and last revisionOctober 15, 2015.

11.2 Assignment 28.

- Geometry
- *Lay: Convex Sets and their Applications*
- Last revision October 15, 2015

Problems

1.9. Prove that any finite set is compact.

1.10. For each of the statement below, either prove it is true or show it is false by a counterexample.
(a) If A is open, then for any set B, $A + B$ is open.
(b) If A and B are both closed, then $A + B$ is closed.

1.11. Verify the following:
(a) A function $f : E^n \to E^m$ is continuous on E^n if and only if for each x in E^n and for each $\epsilon > 0$ there exists a $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$.
(b) If f is continuous and x_k is a sequence which converges to x, then $f(x_k)$ converges to $f(x)$.

1.12. Show that $f : R \to E^n$ defined by $f(\lambda) = \lambda x + (1 - \lambda)y$ is continuous for any fixed $x, y \varepsilon E^n$.

1.13. Let p be a fixed point in E^n. Prove that the function $f : E^n \to R$ defined by $f(x) \equiv d(x, p)$ is continuous.

1.14. Let p be a fixed point in E^n, $p = (x_1, x_2, \ldots x_n)$, $x_1^2 + x_2^2 \ldots x_n^2 > 0$. Prove that the function defined by $f(x) \equiv \langle x, p \rangle$ is continuous.

1.15. Let $f : E^n \to E^m$ be a continuous function and suppose A is a compact subset of E^n. Then $f(A) \equiv \{f(x) : x \varepsilon A\}$ is a compact subset of E^m.

11.3 Assignment 29.

Summary

- Power Series
- *Henrici, Elemente der Mathematik 378*
- Last revision October 15, 2015

Problems

inverse of a certain formal power series

0)

unknowns, say arbitrary complex numbers

$$\sigma_1, \sigma_2 \ldots$$

1)

definition

$$v(t) = 1 - \sigma_1 t + \sum_{\nu=2}^{\infty} \begin{vmatrix} \sigma_1 & 1 & & & \\ \sigma_2 & \sigma_1 & 2 & & \\ \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \nu - 1 \\ \sigma_\nu & \sigma_\nu & \ldots & \ldots & \sigma_1 \end{vmatrix} \frac{(-t^{)}\nu}{\nu!}$$

2)

definition

$$w(t) = 1 + \sigma_1 t + \sum_{\nu=2}^{\infty} \begin{vmatrix} \sigma_1 & -1 & & & \\ \sigma_2 & \sigma_1 & -2 & & \\ \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & -\nu + 1 \\ \sigma_\nu & \sigma_\nu & \ldots & \ldots & \sigma_1 \end{vmatrix} \frac{t^\nu}{\nu!}$$

3)

claim

$$v(t) * w(t) = 1$$

4)

write

$$w(t) = 1 + \sum_{\nu=1}^{\infty} w_\nu t$$

where w_ν does not depend on t. For example:

$$w_1 = \sigma_1$$

$$w_2 = \frac{1}{2!} \begin{vmatrix} \sigma_1 & -1 \\ \sigma_2 & \sigma_1 \end{vmatrix}$$

$$w_3 = \frac{1}{3!} \begin{vmatrix} \sigma_1 & -1 & \\ \sigma_2 & \sigma_1 & -2 \\ \sigma_3 & \sigma_2 & \sigma_1 \end{vmatrix}$$

$$w_4 = \frac{1}{4!} \begin{vmatrix} \sigma_1 & -1 & & \\ \sigma_2 & \sigma_1 & -2 & \\ \sigma_3 & \sigma_2 & \sigma_1 & -3 \\ \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 \end{vmatrix}$$

.

5)

write

$$\sum_{\nu=1}^{\infty} \sigma_\nu t^{\nu-1} = \sigma(t)$$

6)

claim

$$\frac{\frac{dw(t)}{dt}}{w(t)} = \sigma(t)$$

7)

$$\begin{bmatrix} \sigma_1 & -1 & & & \\ \sigma_2 & \sigma_1 & -2 & & \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix} \begin{bmatrix} 1 \\ w_1 \\ w_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

For example:

$$n = 1$$

$$\sigma_1 - w_1 = 0$$

$$n = 2$$

$$\sigma_2 + \sigma_1 w_1 - 2w_2 = 0$$

$$\sigma_2 + \sigma_1\sigma_1 - 2\frac{1}{2!}\begin{vmatrix} \sigma_1 & -1 \\ \sigma_2 & \sigma_1 \end{vmatrix} = 0$$

$$n = 3$$

$$\sigma_3 + \sigma_2 w_1 + \sigma_1 w_2 - 3w_3 = 0$$

$$\sigma_3 + \sigma_2\sigma_1 + \sigma_1\frac{1}{2!}\begin{vmatrix} \sigma_1 & -1 \\ \sigma_2 & \sigma_1 \end{vmatrix} - 3\frac{1}{3!}\begin{vmatrix} \sigma_1 & -1 & 0 \\ \sigma_2 & \sigma_1 & -2 \\ \sigma_3 & \sigma_2 & \sigma_1 \end{vmatrix} = 0$$

Expansion by the last row gives equation.

$$n = 4$$

$$\sigma_4 + \sigma_3 w_1 + \sigma_2 w_2 + \sigma_1 w_3 - 4w_4 = 0$$

$$\sigma_4 + \sigma_3\sigma_1 + \sigma_2\frac{1}{2!}\begin{vmatrix} \sigma_1 & -1 \\ \sigma_2 & \sigma_1 \end{vmatrix} + \sigma_1\frac{1}{3!}\begin{vmatrix} \sigma_1 & -1 & 0 \\ \sigma_2 & \sigma_1 & -2 \\ \sigma_3 & \sigma_2 & \sigma_1 \end{vmatrix} = 0 \tag{11.1}$$

$$-4\frac{1}{4!}\begin{vmatrix} \sigma_1 & -1 & & \\ \sigma_2 & \sigma_1 & -2 & \\ \sigma_3 & \sigma_2 & \sigma_1 & -3 \\ \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 \end{vmatrix} = 0$$

Expansion by the last row gives equation. Recursion for the general case.

8)

$$\begin{vmatrix} \sigma_1 & -1 & & & \\ \sigma_2 & \sigma_1 & -2 & & \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & -\nu+1 \\ \sigma_\nu & \sigma_{\nu-1} & \dots & \dots & \sigma_1 \end{vmatrix} : \begin{vmatrix} 1 & & & & \\ & 2 & & & \\ & & \dots & & \\ & & & \dots & \\ & & & & \nu \end{vmatrix} = w_\nu$$

9)

$$\ln w(t) = \int_0^t \sigma(\tau) d\tau$$

10)

$$w(t) = \exp\left(\int_0^t \sigma(\tau) d\tau\right)$$

11)

$$w^{-1}(t) = \exp\left(-\int_0^t \sigma(\tau) d\tau\right) = v(t)$$

11.4 Assignment 30.

Summary

- Higher Arithmetic
- *Davenport-Guy*
- Last revision October 15, 2015

Problems

Gaussian integers Prove that Gaussian integers have unique factorization.

11.5 Assignment 22.

Summary

- Determinants and Quadratic Forms
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision October 15, 2015

Problems

Inverse of Hilbert matrix Let $\{a_1, a_2, \ldots, a_n, b_1, \ldots, b_n\}$ be a set of $2n$ *distinct* real numbers on an interval I. Define an $n \times n$ matrix H whose (i, j) entry is

$$h_{ij} = \frac{1}{a_i - b_j}.$$

Let $p_1, p_2, \ldots, p_n$ be the Lagrange interpolation polynomials for $\{a_1, a_2, \ldots, a_n\}$, that is the unique polynomials of degree less than or equal to $n-1$ such that

$$p_i(a_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots n,$$

where δ_{ij} is the Kronecker delta, explicitely

$$p_i(x) = \prod_{k=1, k\neq i}^{n} \frac{(x-a_k)}{(a_i - a_k)} = \frac{(x-a_1)\ldots(x-a_{i-1})(x-a_{i+1})\ldots(x-a_n)}{(a_i-a_1)\ldots(a_i-a_{i-1})(a_i-a_{i+1})\ldots(a_i-a_n)}.$$

Write $p(x) = \prod_{k=1}^{n}(x - a_k)$, then

$$p'(x) = \sum_{i=1} \prod_{k=1, k\neq i}^{n} (x - a_k),$$

$$p'(a_i) = (a_i - a_1)\ldots(a_i - a_{i-1})(a_i - a_{i-1})\ldots(a_i - a_n)$$

and

$$p_i(x) = \frac{p(x)}{p'(a_i)(x - a_i)}$$

which is the conventional notation. Recall that

$$P^{(n-1)}_{a_1, a_2, \ldots a_n} = \{p_i, \ i = 1, \ldots, n\}$$

is called the *Lagrange basis*, a collection of n special polynomials of degree $(n-1)$ which are 0 or 1 at nodes $a_1, a_2, \ldots a_n$. It is customary to assume

$$\text{either } a_1 < a_2 < \ldots < a_n; \text{ or } a_1 > a_2 > \ldots > a_n;$$

and to suppress the dependence of p_i on x and nodes. Further, if f is a polynomial then the linear combination

$$\sum_{i=1}^{n} f(a_i) p_i$$

approximates f, in fact, agrees with f at the nodes. If f is a polynomial of degree $n-1$ or less then

$$f = \sum_{i=1}^{n} f(a_i) p_i.$$

Similarly, since b_j's are distinct, let $q_1, \ldots, q_n$ be the Lagrange interpolation polynomials for $\{b_1, \ldots, b_n\}$. That is, they are the unique polynomials of degree not exceeding $n-1$ with the property that

$$q_i(b_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots n.$$

Write $q(x) = \prod_{k=1}^{n}(x - b_k)$, $q = q(x)$, then these polynomials are defined explicitely by

$$q_j(x) = \prod_{l=1, l \neq j}^{n} \frac{(x - b_l)}{(b_j - b_l)} = \frac{q}{q'(b_j)(x - b_j)}.$$

These polynomial have the property that, for any polynomial f of degree not exceeding $n-1$, we have

$$f = \sum_{j=1}^{n} f(b_j) q_j.$$

Claim: Let matrix H be defined as above

$$H = [h_{ij}], \quad h_{ij} = \frac{1}{a_i - b_j}.$$

Then H is invertble, $G = H^{-1}$,

$$G = [g_{ij}], \quad g_{ij} = (a_j - b_i) p_j(b_i) q_i(a_j).$$

Proof: Write $f = p(b_k)q_k$, f is a polynomial of degree $n-1$, where

$$p(b_k) = \prod_{l=1}^{n}(b_k - a_l), \quad q_k = \prod_{l=1, l\neq k}^{n} \frac{(x-b_l)}{(b_k - b_l)}.$$

Then

$$p(b_k)q_k = \sum_{i=1}^{n} p(b_k)q_k(a_i)p_i.$$

Multiplying both sides by $\frac{1}{p}$ and using the convention on p_i

$$\frac{p(b_k)q_k}{p} = \sum_{i=1}^{n} p(b_k)q_k(a_i)\frac{p_i}{p} = \sum_{i=1}^{n} p(b_k)q_k(a_i)\frac{1}{p'(a_i)(x-a_i)}.$$

Set $x = b_j$, $p(b_j) \neq 0$, because $a_i \neq b_j$ for $i, j = 1, 2, \ldots, n$

$$\frac{p(b_k)q_k(b_j)}{p(b_j)} = \sum_{i=1}^{n} p(b_k)q_k(a_i)\frac{1}{p'(a_i)(b_j - a_i)}.$$

Now, we are ready to make crucial observations. If $j = k$ then the left-hand side is equal to 1. On the other hand, if $j \neq k$ then the left-hand side vanishes because $q_k(b_j) = 0$ by construction. Moreover,

$$\frac{1}{(b_j - a_i)} = -h_{ij},$$

so, on the right hand side, we have a scalar product of two vectors, we call them tentatively $\vec{u_k}$ and $\vec{v_j}$,

$$\vec{u_k} = \left[-\frac{p(b_k)q_k(a_1)}{p'(a_1)}, -\frac{p(b_k)q_k(a_2)}{p'(a_2)}, \ldots, -\frac{p(b_k)q_k(a_n)}{p'(a_n)}\right]$$

$$\vec{v_j} = \left[\frac{1}{(b_j - a_1)}, \frac{1}{(b_j - a_2)}, \ldots, \frac{1}{(b_j - a_n)}\right].$$

Let $\vec{u_k}$ be the $k-th$ row of $H^{-1} = G$, and $\vec{v_j}$ the $j-th$ column of H then

$$(\vec{u_k}, \vec{v_j}) = \delta_{kj}$$

and G is the inverse of H. Moreover,

$$\frac{p}{p'(a_i)} = (x - a_i)p_i$$

and

$$\frac{p(b_k)}{p'(a_i)} = (b_k - a_i)p_i(b_k)$$

so

$$g_{ij} = (a_i - b_k)p_i(b_k)q_k(a_i)$$

which proves the claim (with some indeces relabeled).

Let $a_i = i$ and $b_j = 1 - j$. Define an $n \times n$ *Hilbert-matrix* $H_n = [h_{ij}]$ by

$$h_{ij} = \frac{1}{i+j-1}.$$

Claim: The *Hilbert-matrix* is invertible, with inverse given by $G_n = [g_{ij}]$,

$$g_{i,j} = (j+1-1)\left[\prod_{k=1,\ k\neq j}^{n} \frac{(1-i-k)}{(j-k)}\right]\left[\prod_{l=1,\ l\neq i}^{n} \frac{(j+l-1)}{(l-i)}\right].$$

Proof: First, we factor out a (-1) from each of the $(n-1)$ factors of the numerator of the first product, and from the $(n-1)$ factors of the denominator of the second product, to obtain

$$g_{i,j} = (j+1-1)\left[\prod_{k=1,\ k\neq j}^{n} \frac{(i+k-1)}{(j-k)}\right]\left[\prod_{l=1,\ l\neq i}^{n} \frac{(j+l-1)}{(i-l)}\right].$$

Let us simplify the first expression in square braces:

$$\prod_{k=1,\ k\neq j}^{n} \frac{(i+k-1)}{(j-k)} =$$

$$= \frac{(i)(i+1)\dots(i+n-1)}{(i+j-1)} \frac{1}{(j-1)(j-2)\dots(j-(j-1))(j-(j+1))\dots(j-n)}$$

$$= \frac{1}{(i+j-1)} \frac{(n+i-1)!}{(i-1)!} \frac{1}{(j-1)!} \frac{1}{(j-(j+1))\dots(j-(j+n-j))}$$

$$= \frac{(-1)^{n-j}(n+i-1)!}{(i+j-1)(i-1)!(j-1)!((j+1)-j)\dots((j+n-j)-j)}$$

$$= \frac{(-1)^{n-j}(n+i-1)!}{(i+j-1)(i-1)!(j-1)!((j+1)-j)\dots((j+n-j)-j)}$$

$$= \frac{(-1)^{n-j}(n+i-1)!}{(i+j-1)(i-1)!(j-1)!(n-j)!}.$$

The second expression in square braces is identical to the first after interchanging i and j. Thus the equation

$$g_{i,j} = (i+j-1)\left[\frac{(-1)^{n-j}(n+i-1)!}{(i+j-1)(i-1)!(j-1)!(n-j)!}\right]\left[\frac{(-1)^{n-i}(n+j-1)!}{(i+j-1)(i-1)!(j-1)!(n-i)!}\right],$$

simplifies,

$$(-1)^{n-j}(-1)^{n-i} = (-1)^{2n-j-i} = (-1)^{2n-(j+i)} = (-1)^{j+i}$$

because only the parity of the exponent counts, moreover, we can factor out the duplicate terms

$$\frac{(-1)^{n-j}(n+i-1)!}{(i+j-1)^2(i-1)!^2(j-1)!^2}.$$

$$\begin{aligned} g_{i,j} &= (-1)^{j+i}(i+j-1)\left(\frac{(n+i-1)!(i+j-1)!}{(n-j)!(i+j-1)!}\right)\left(\frac{(n+j-1)!(i+j-1)!}{(n-i)!(i+j-1)!}\right) \\ &\times \left(\frac{1}{(i+j-1)^2(i-1)!^2(j-1)!^2}\right) \end{aligned}$$

$$\binom{n+i-1}{n-j} = \frac{(n+i-1)!}{(n-j)!(i+j-1)!}$$

$$\binom{n+j-1}{n-i} = \frac{(n+j-1)!}{(n-i)!(i+j-1)!}$$

$$\begin{aligned} g_{i,j} &= (-1)^{j+i}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i} \\ &\times \left(\frac{(i+j-2)!(i+j-2)!}{(i-1)!(j-1)!(i-1)!(j-1)!}\right) \end{aligned}$$

$$g_{i,j} = (-1)^{j+i}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}\binom{i+j-2}{i-1}$$

11.6 Trick or Treat.

Summary

- Determinants and Quadratic Forms
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision October 15, 2015

Problems

VII 3, Cauchy

$$\left|\frac{1}{a_\lambda + b_\mu}\right|_1^n = \frac{\Pi_{j>k}^{1,2,\dots n}(a_j - a_k)(b_j - b_k)}{\Pi_{\lambda,\mu}^{1,2,\dots n}(a_\lambda + b_\mu)}$$

Case $n = 2$:

$$\left|\frac{1}{a_\lambda + b_\mu}\right|_1^2 = \frac{(a_2 - a_1)(b_2 - b_1)}{(a_2 + b_1)(a_2 + b_2)(a_1 + b_1)(a_1 + b_2)}$$

$$\begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} \end{vmatrix} = \begin{vmatrix} \frac{1}{(a_1+b_1)} - \frac{1}{(a_2+b_1)} & \frac{1}{(a_1+b_2)} - \frac{1}{(a_2+b_2)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} \end{vmatrix} =$$

$$\begin{vmatrix} \frac{(a_2+b_1)-(a_1+b_1)}{(a_2+b_1)(a_1+b_1)} & \frac{(a_1+b_2)-(a_2+b_2)}{(a_2+b_2)(a_1+b_2)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} \end{vmatrix} =$$

$$\begin{vmatrix} \frac{a_2-a_1}{(a_2+b_1)(a_1+b_1)} & \frac{a_2-a_1}{(a_1+b_2)(a_2+b_2)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} \end{vmatrix} = \frac{a_2-a_1}{(a_2+b_1)(a_2+b_2)} \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} \\ 1 & 1 \end{vmatrix}$$

$$= \frac{a_2-a_1}{(a_2+b_1)(a_2+b_2)}\left(\frac{1}{(a_1+b_1)} - \frac{1}{(a_1+b_2)}\right) = \frac{a_2-a_1}{(a_2+b_1)(a_2+b_2)}\left(\frac{b_2-b_1}{(a_1+b_1)(a_1+b_2)}\right)$$

$$= \frac{(a_2-a_1)(b_2-b_1)}{(a_1+b_1)(a_1+b_2)(a_2+b_1)(a_2+b_2)} = \Delta_2. \checkmark$$

Case $n = 3$:

$$\Delta_3 = \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & \frac{1}{(a_1+b_3)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & \frac{1}{(a_2+b_3)} \\ \frac{1}{(a_3+b_1)} & \frac{1}{(a_3+b_2)} & \frac{1}{(a_3+b_3)} \end{vmatrix} =$$

Subtract the $n-th$ row from the first $n-1$ rows.

$$\begin{vmatrix} \frac{1}{(a_1+b_1)} - \frac{1}{(a_3+b_1)} & \frac{1}{(a_1+b_2)} - \frac{1}{(a_3+b_2)} & \frac{1}{(a_1+b_3)} - \frac{1}{(a_3+b_3)} \\ \frac{1}{(a_2+b_1)} - \frac{1}{(a_3+b_1)} & \frac{1}{(a_2+b_2)} - \frac{1}{(a_3+b_2)} & \frac{1}{(a_2+b_3)} - \frac{1}{(a_3+b_3)} \\ \frac{1}{(a_3+b_1)} & \frac{1}{(a_3+b_2)} & \frac{1}{(a_3+b_3)} \end{vmatrix} =$$

Then by

$$\frac{1}{(a_k+b_i)} - \frac{1}{(a_3+b_i)} = \frac{a_3 - a_k}{(a_k+b_i)(a_3+b_i)}; k = 1, 2;\ i = 1, 2, 3$$

$$\begin{vmatrix} \frac{(a_3-a_1)}{(a_1+b_1)(a_3+b_1)} & \frac{(a_3-a_1)}{(a_1+b_2)(a_3+b_2)} & \frac{(a_3-a_1)}{(a_1+b_3)(a_3+b_3)} \\ \frac{(a_3-a_2)}{(a_2+b_1)(a_3+b_1)} & \frac{(a_3-a_2)}{(a_2+b_2)(a_3+b_2)} & \frac{(a_3-a_2)}{(a_2+b_3)(a_3+b_3)} \\ \frac{1}{(a_3+b_1)} & \frac{1}{(a_3+b_2)} & \frac{1}{(a_3+b_3)} \end{vmatrix} =$$

$$\frac{(a_3-a_1)(a_3-a_2)}{(a_3+b_1)(a_3+b_2)(a_3+b_3)} \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & \frac{1}{(a_1+b_3)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & \frac{1}{(a_2+b_3)} \\ 1 & 1 & 1 \end{vmatrix} = \Delta_3.$$

Next, subtract the $n-th$ column from the first $n-1$ columns.

$$\Delta_3 = [\frac{(a_3-a_1)(a_3-a_2)}{(a_3+b_1)(a_3+b_2)(a_3+b_3)} \begin{vmatrix} \frac{1}{(a_1+b_1)} - \frac{1}{(a_1+b_3)} & \frac{1}{(a_1+b_2)} - \frac{1}{(a_1+b_3)} & \frac{1}{(a_1+b_3)} \\ \frac{1}{(a_2+b_1)} - \frac{1}{(a_2+b_3)} & \frac{1}{(a_2+b_2)} - \frac{1}{(a_2+b_3)} & \frac{1}{(a_2+b_3)} \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_3 = [\frac{(a_3-a_1)(a_3-a_2)}{(a_3+b_1)(a_3+b_2)(a_3+b_3)}\begin{vmatrix} \frac{b_3-b_1}{(a_1+b_1)(a_1+b_3)} & \frac{b_3-b_2}{(a_1+b_2)(a_1+b_3)} & \frac{1}{(a_1+b_3)} \\ \frac{b_3-b_1}{(a_2+b_1)(a_2+b_3)} & \frac{b_3-b_2}{(a_2+b_2)(a_2+b_3)} & \frac{1}{(a_2+b_3)} \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_3 = [\frac{(a_3-a_1)(a_3-a_2)(b_3-b_1)(b_3-b_2)}{(a_1+b_3)(a_3+b_1)(a_3+b_2)(a_3+b_3)(a_2+b_3)}\begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & 1 \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_3 = \frac{(a_3-a_1)(a_3-a_2)(b_3-b_1)(b_3-b_2)}{(a_1+b_3)(a_3+b_1)(a_3+b_2)(a_3+b_3)(a_2+b_3)}\Delta_2$$

Reduction to $n = 2$.

$$\Delta_3 = \frac{(a_3-a_1)(a_3-a_2)(b_3-b_1)(b_3-b_2)}{(a_1+b_3)(a_3+b_1)(a_3+b_2)(a_3+b_3)(a_2+b_3)}\frac{(a_2-a_1)(b_2-b_1)}{(a_1+b_1)(a_1+b_2)(a_2+b_1)(a_2+b_2)}\surd$$

The general case is as follows. Consider

$$\Delta_n = \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & \cdots & \frac{1}{(a_1+b_n)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & \cdots & \frac{1}{(a_2+b_n)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(a_n+b_1)} & \frac{1}{(a_n+b_2)} & \cdots & \frac{1}{(a_n+b_n)} \end{vmatrix}.$$

Subtract the last row from the preceeding rows and take out the following factors from the columns

$$\frac{1}{(a_n+b_1)}, \frac{1}{(a_n+b_2)}, \cdots \frac{1}{(a_n+b_{n-1})}, \frac{1}{(a_n+b_n)}$$

and the factors

$$(a_n-a_1), (a_n-a_2), \ldots (a_n-a_{n-1}), 1,$$

from the rows inthe same manner as in case $n = 3$. Write

$$C_1^n = \frac{1}{(a_n+b_1)} \times \frac{1}{(a_n+b_2)} \times \ldots \times \frac{1}{(a_n+b_{n-1})} \times \frac{1}{(a_n+b_n)}$$

and

$$C_2^n = (a_n - a_1) \times (a_n - a_2 \times \ldots \times (a_n - a_{n-1}) \times 1.$$

Mutatis mutandis for the columns: in the remaining determinant subtract the last column from the preceeding columns and factor out

$$(b_n - b_1), (b_n - b_2), \ldots (b_n - b_{n-1}), 1$$

and

$$\frac{1}{(a_1 + b_n)}, \frac{1}{(a_2 + b_n)}, \ldots \frac{1}{(a_{n-1} + b_n)}, \frac{1}{(a_n + b_n)},$$

respectively. Write

$$C_3^n = (b_n - b_1) \times (b_n - b_2) \times \ldots \times (b_n - b_{n-1})$$

and

$$C_4^n = \frac{1}{(a_1 + b_n)} \times \frac{1}{(a_2 + b_n)} \times \ldots \frac{1}{(a_{n-1} + b_n)} \times \frac{1}{(a_n + b_n)}.$$

There remains a $(n-1)$-rowed corner minor of the given determinant

$$\Delta_n = C_1^n C_2^n C_3^n C_4^n \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & \ldots & \frac{1}{(a_1+b_{n-1})} & 1 \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & \ldots & \frac{1}{(a_2+b_{n-1})} & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(a_{n-1}+b_1)} & \frac{1}{(a_{n-1}+b_2)} & \ldots & \frac{1}{(a_{n-1}+b_{n-1})} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Mathematical induction completes the proof.

Define the Hilbert matrix, calculate the determinant, write out the inverse (Choi, AMM, Vol 90, No. 5, May 1983, pp 301-312). Here comes a very interesting application of the result above, attributed to Cauchy. As is well known, the *Hilbert - matrix* is defined by

$$A_n = \left[\frac{1}{i + j - 1}\right]_{i,j=1}^{i,j=n}$$