Pinter Consulting
Progress Report Q4
New Series No. 6.

J K Pinter, Dr.Tech.

January 2, 2015

Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

Előszó

Dr. No - vidám és élénk mint egy ifjú. Nyílt, gondolkodásra termett homloka elpusztíthatatlan derültség és öröm székhelye, a gondolatokban leggazdagabb beszéd ömlik ajkairól; tréfa, elmésség és hangulat a rendelkezésére állanak és tanító előadása a legszórakoztatóbb tásaság.

\- - - -

Introduction

Pinter Consulting of Calgary, Alberta practices Mathematics, promotes clear thinking and offers Consultations, Tutorials and Seminars in Mathematics.

Contents

Chapter 6

Proceedings

6.1 Summary of Current Report

Private study for professional development:

Records of activities at Pinter Consulting : no extracurricular activities.

Collection of problems with our own solutions: Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis, Part One, Chap. 1, Additive Number Theory, Combinatorial Problems 1-23, and Applications, with a Short Essay, revised and corrected.

Socratic Programme

- Analysis
- Algebra and Number Theory
- Geometry
- Differential and Integral Equations

Continuos improvement, corrections and last revision January 2, 2015.

6.2 A Short Essay on Combinatorial Problems

Summary

- Combinatorial Analysis
- *Pólya - Szegő; Riordan*
- Last revision January 2, 2015

Formal Power Series

Let F denote the *field of real numbers* and let x be an *indeterminate* . Then the (formal) infinite sum with $a_i \epsilon F, \forall \imath, \imath \geq 0$ is called a *(formal) power series*

$$S(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

This definition includes polynomials when $a_i = 0, \imath > M, M \geq 0$. We call the object a *formal* power series because we do not assign any value to this infinite sum. Let

$$T(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots$$

be another *(formal) power series* . $S(x) = T(x)$ if and only if

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2, \ldots a_n = b_n \ldots;\ \forall n$$

Further, *addition* and *multiplication* are defined for power series by

$$S(x) + T(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n + \ldots$$

and

$$S(x)T(x) = (a_0 b_0) + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 +$$

$$\ldots + (\sum_{i+j=n} a_i b_j) x^n + \ldots,$$

respectively. Write

$$V(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots.$$

Then the equalities

$$S(x) + T(x) = T(x) + S(x)$$

$$S(x)T(x) = T(x)S(x)$$

$$(S(x) + T(x)) + V(x) = S(x) + (T(x) + V(x))$$

$$(S(x)T(x))V(x) = S(x)(T(x)V(x))$$

$$(S(x)(T(x) + V(x)) = S(x)T(x) + S(x)V(x)$$

are all valid because the coefficients are taken from F, the field of real numbers. Subtraction, the inverse of addition, is defined as well

$$S(x) - T(x) = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \ldots + (a_n - b_n)x^n + \ldots$$

Therefore the set of all formal power series form a *commutative ring.*Let us list the *rules of computation* in the ring:

1. (i) + is defined, (ii) + is commutative, (iii) + is associative, (iv) + has an inverse;

2. (i) $\times$ is defined, (ii) $\times$ is commutative, (iii) $\times$ is associative;

3. $\times$ is distributive over +;

In fact, we have a *module*

$$rS(x) = ra_0 + ra_1x + ra_2x^2 + \ldots + ra_nx^n + \ldots$$

There is a *unity element* for multiplication and a *zero element* for addition. Some (formal) power series can have an inverse:

$$S(x)T(x) = V(x)$$

$$(a_0 + a_1x + a_2x^2 + \ldots)(b_0 + b_1x + b_2x^2 + \ldots) = c_0 + c_1x + c_2x^2 + \ldots$$

or

$$\begin{aligned} a_0b_0 &= c_0 \\ a_1b_0 + a_0b_1 &= c_1 \\ a_2b_0 + a_1b_1 + a_0b_2 &= c_2 \\ a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3 &= c_3 \\ &\ldots \end{aligned}$$

If $S(x), V(x)$ are given and $a_0 \neq 0$ then $T(x)$ can be determined by *recursion*

$$b_0 = \frac{c_0}{a_0}, \quad b_1 = a_0^{-2} \det \begin{bmatrix} a_0 & c_0 \\ a_1 & c_1 \end{bmatrix}, \dots$$

or by solving systems of linear equations. For example, to find b_3 we need to solve

$$\begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

By Cramer's rule

$$b_3 = \frac{\det \begin{bmatrix} a_0 & 0 & 0 & c_0 \\ a_1 & a_0 & 0 & c_1 \\ a_2 & a_1 & a_0 & c_2 \\ a_3 & a_2 & a_1 & c_3 \end{bmatrix}}{\det \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}} = a_0^{-4} \det \begin{bmatrix} a_0 & 0 & 0 & c_0 \\ a_1 & a_0 & 0 & c_1 \\ a_2 & a_1 & a_0 & c_2 \\ a_3 & a_2 & a_1 & c_3 \end{bmatrix}.$$

By setting $V(x)$ to be the unity element for multiplication

$$S^{-1}(x) = T(x).$$

Note also

$$S(x)T(x) = 0 \Rightarrow S(x) = 0 \vee T(x) = 0$$

that is the commutative ring of (formal) power series is an *integral domain* .

Convolution of Formal Power Series

The multiplication of formal power series is also known as *faltung* or *convolution.* Consider

$$S(x)T(x) = (a_0b_0) + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 +$$

$$\dots + (\sum_{i+j=n} a_ib_j)x^n + \dots,$$

We can obtain this result more easily by and assigning to $S(x)$ a certain lower triangular matrix of infinite dimension

$$\mathbf{A} = \begin{bmatrix} a_0 & 0 & 0 & 0 & \dots \\ a_1 & a_0 & 0 & 0 & \dots \\ a_2 & a_1 & a_0 & 0 & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

and assigning to $T(x)$ a vector of infinite dimension

$$\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \dots \end{bmatrix}.$$

Then we multiply matrix $\mathbf{A}$ by vector $\mathbf{b}$ on the right

$$\mathbf{Ab} = \mathbf{c}$$

to obtain the resultant vector

$$\mathbf{c} = \begin{bmatrix} a_0 b_0 \\ a_1 b_0 + a_0 b_1 \\ a_2 b_0 + a_1 b_1 + a_0 b_2 \\ a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3 \\ \dots \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \dots \end{bmatrix}.$$

Principles of Combinatorial Analysis

Combinatorial Analysis (a.k.a. Combinatorics) is mainly concerned with problems of discrete sets, such as enumeration of subsets satisfying certain conditions. The analysis and the construction of discrete sets are much more difficult in combinatorial analysis than those of infinite sets in real analysis where we have topology at our disposal.

The most fundamental principles in combinatorial analysis are the following three principles. Let Ω be a finite set and let $|A|$ denote the number of elements in a subset A of Ω.

1. **Rule of sums:** If $A \cap B = \emptyset$, then $|A + B| = |A| + |B|$.

2. **Rule of products:** $|A \times B| = |A| \times |B|$, where $A \times B$ is the direct product of two sets A and B.

3. **Principle of inclusion-exclusion:**

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots \cup A_n| &= \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\ &\quad - \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|. \end{aligned}$$

The **Rule of sums** can be stated (after Riordan) like this: If an object A may be chosen in m ways and B in n ways, "either A or B" may be chosen in $m+n$ ways. The **Rule of products:** If an object A may be chosen in m ways, and thereafter B in n ways, both "A and B" may be chosen in this order in mn ways. **Principle of inclusion-exclusion:** If of N objects $N(a)$ have property a, $N(b)$ b, ..., $N(ab)$ both a and b, ..., $N(abc)$ a, b and c, and so on, the number $N(a'b'c'\ldots)$ with none of these properties is given by

$$\begin{aligned} N(a'b'c'\ldots) &= N[(1-a)(1-b)(1-c)\ldots] \\ &= N[1-a-b-c\ldots+ab+ac+\ldots-abc\ldots] \\ &= N - N(a) - N(b) - N(c)\ldots \\ &\quad + N(ab) + N(ac)\ldots + N(bc)\ldots \\ &\quad - N(abc) \\ &\quad \ldots. \end{aligned}$$

Here we used a symbolic form $a' = 1 - a$ for the *complement* of a.

Generators

Generators or generating enumerator functions are great computational devices in combinatorial analysis. Let us start with two *prototype generators* . The first is

$$(1+t)^n = \sum_{k=0}^{k=n} \binom{n}{k} t^k = \sum_{k=0}^{k=n} a_k t^k$$

where

$a_k =$ number of **unordered** selection of k objects out of n unlike objects .

Let us check this by setting $n = 3$ with $\{a, b, c\}$ as the set of 3 unlike objects.

$$(1 + t)^3 = 1 + 3t^1 + 3t^2 + 1t^3$$

Clearly, $a_0 = 1$ by convention, $a_1 = 3$ because there are 3 choices to make a selection of 1 element, namely $\{a\}, \{b\}, \{c\}$. There are 3 choices to make a unordered selection of 2 elements: $\{ab\}, \{ac\}, \{bc\}$. Thus $a_2 = 3$. There is only one unordered selection of all three elements : $a_3 = 1$. Therefore the coefficients of the expansion are the ordered set of the numbers of unordered selections of k objects out of n unlike objects, $k = 0, 1, 2 \ldots n$.

Next, let us examine the second prototype generator:

$$(1 + t)^n = \sum_{k=0}^{k=n} \binom{n}{k} t^k = \sum_{k=0}^{k=n} \frac{a_k}{k!} t^k$$

where

$a_k =$ number of **ordered** selection of k objects out of n unlike objects .

Again $a_0 = 1$ by convention, $a_1 = 3 \times 1! = 3$; $a_2 = 3 \times 2! = 6$. Indeed there are 6 different ways to select two elements out of three with regard to order. These are $\{ab\}, \{ac\}, \{ba\}, \{bc\}, \{ca\}, \{cb\}$. There are 6 different ways how the three unlike objects can be arranged: $\{abc\}, \{acb\}, \{bac\}, \{bca\}, \{cab\}, \{cba\}$, thus $a_3 = 1 \times 3! = 6$. After this illustration we are ready to define

1. *Ordinary generating function*

$$A(t) = \sum_{k=0}^{k=\infty} a_k t^k,$$

2. *Exponential generating function*

$$E(t) = \sum_{k=0}^{k=\infty} \frac{a_k}{k!} t^k.$$

The variable, or indeterminate t need not be defined, these two generating functions are members of the commutative ring of (formal) power series. However if t is taken as a real or complex number then we have to examine *convergence* . If the sequence of a_k-s are bounded in the definition of $A(t)$

then the sum for $A(t)$ converges for $|t| < 1$. The sum for $E(t)$ converges for all t.

It is possible to develop a *calculus of generating function* , and to explore their connection to *analytic functions* , but we will not do that here. Instead, we will apply formal operations such as the ones we already discussed along with termwise differentiation and integration if we need them and we will justify our methods by heuristic arguments.

Some Simple Generators

Here are some simple generators listed in Riordan's book.

a_k	$A(t)$	$E(t)$
a^k	$(1-at)^{-1}$	$\exp(at)$
k	$t(1-t)^{-2}$	$t\exp(t)$
$k(k-1)$	$2t^2(1-t)^{-3}$	$t^2\exp(t)$
k^2	$t(t+1)(1-t)^{-3}$	$t(t+1)\exp(t)$

Diophantine Equations

A necessary and sufficient condition for the equation

$$ax + by = n$$

to have integer solution in x, y is that the graetest common factor of a, b divide n. In particular, if a, b are relative prime numbers then the equation has an integer solution in x, y. Consider

$$21x - 17y = 2.$$

$$|17| < |21|;\ y = \frac{21x-2}{17} = x + \frac{4x-2}{17} = x + u$$

$$u = \frac{4x-2}{17} \text{ integer}$$

$$4x - 17u = 2,\ |4| < |17|,\ x = \frac{17u+2}{4} = 4u + \frac{u+2}{4} = 4u + v,$$

$$v = \frac{u+2}{4} \text{ integer}$$

$u = 4v - 2$

$x = 4u + v = 4(4v - 2) + v = 16v - 8 + v = 17v - 8.$

$y = x + u = (17v - 8) + (4v - 2) = 21v - 10.$

Therefore each set of solution is given by

$x = 17v - 8,\ y = 21v - 10;\ v = \ldots, -2, -1, 0, 1, 2, \ldots$

Note that the algorithm terminated in finite steps:

$|21| > |17| > |4| > 0.$

The *typical question* in the first Chapter is as follows: What is the number of solutions to

$x + 2y + 3z = n,\ x, y, z$ non-negative integers

The required number is the coefficient a_k of t^k in the power series expansion for

$$\frac{1}{(1-t)(1-t^2)(1-t^3)} = \sum_{k=0}^{k=\infty} a_k t^k.$$

The power series can be obtained by the method of *partial fractions* .

Partial Fractions

Every polynomial

$g(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

may be decomposed into *irreducible factors*

$g(x) = a_0 p_1^{k_1} p_2^{k_2} \ldots p_l^{k_l} q_1^{k_{l+1}} q_2^{k_{l+2}} \ldots q_{l+m}^{k_{l+m}}$

where $p_i, i = 1, 2, \ldots l$, are irreducible linear polynomials with leading coefficients equal to unity, and $q_j, j = 1, 2, \ldots m$, are irreducible quadratic polynomials with similar leading coefficients. The linear polynomials $p_i(x) = x - \alpha_i$ correspond to the real roots of $g(x), g(\alpha_i) = 0$, whereas the quadratic polynomials $q_j(x) = (x - \beta_j)(x - \bar{\beta}_j = x^2 - (\beta_j + \bar{\beta}_j)x + \beta_j \bar{\beta}_j$ correspond to pairs of complex conjugate roots $g(\beta_j) = g(\bar{\beta}_j) = 0$. The numbers $k_1, k_2 \dots k_l$ and $k_{l+1}, k_{l+2} \dots k_{l+m}$ denote the multiplicity of roots:

$$k_1 + k_2 + \dots + k_l + 2(k_{l+1} + k_{l+2} \dots k_{l+m}) = n,$$

guaranteed by the *Fundamental Theorem of Algebra* . This decomposition is essentially unique and can be adapted easely when one or more components are missing. For $n = 0, n = 1$, respectively

$$g_0(x) = const = a_0; \quad g_1(x) = ax + b = a(x + \frac{b}{a}).$$

A *rational function* is *proper* if the degree of the numerator is less than the degree of the denominator:

$$R(x) = \frac{f(x)}{g(x)} = \frac{b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0}{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0}, \ m < n$$

Every proper rational function has a unique decomposition into a sum *partial fractions* . Each partial fraction has for denominator a power of some irreducible factor. For example

$$f(x) = 2x^4 - 10x^3 + 7x^2 + 4x + 3$$

$$g(x) = x^5 - 2x^3 + 2x^2 - 3x + 2 = (x+2)(x-1)^2(x^2+1).$$

$$R(x) = \frac{2x^4 - 10x^3 + 7x^2 + 4x + 3}{x^5 - 2x^3 + 2x^2 - 3x + 2} = \frac{2x^4 - 10x^3 + 7x^2 + 4x + 3}{(x+2)(x-1)^2(x^2+1)}$$

$$R(x) = \frac{A}{x+2} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$$

where numbers A, B, C, D, E are to be determined. Write

$$g(x) = s_1(x)(x+2) = s_2(x)(x-1)^2 = s_3(x)(x-1) = s_4(x)(x^2+1).$$

Then

$$R(x) = \frac{As_1(x)}{(x+2)s_1(x)} + \frac{Bs_2(x)}{(x-1)^2 s_2(x)} + \frac{Cs_3(x)}{(x-1)s_3(x)} + \frac{(Dx+E)s_4}{(x^2+1)s_4}$$

or

$$R(x) = \frac{As_1(x) + Bs_2(x) + Cs_3(x) + (Dx+E)s_4}{g(x)}.$$

Thus

$$As_1(x) + Bs_2(x) + Cs_3(x) + (Dx+E)s_4 = f(x)$$

and A, B, C, D, E can be determined after collecting like terms.

Binomial Theorem and Series

$$(a+b)^n = \sum_{k=0}^{k=n} \binom{n}{k} a^k b^{n-k}; \quad \binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}; \quad \text{n,k integers}$$

$$(1+x)^\alpha = \sum_{k=0}^{k=\infty} \binom{\alpha}{k} x^k; \quad (-1 < x < 1)$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!}; \quad \alpha \text{ real}, k \text{ integer}$$

Expansions :

$$(1+z)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 +$$

$$\ldots + \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!}x^k + \ldots$$

in particular,

$$(1+z)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots +$$

Denumerants

Given positive integers

$$a_1, a_2, \ldots a_l$$

$$p_1, p_2, \ldots p_l$$

find E_n, the number of solutions to the equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_l x_l = n$$

subject to conditions

$$(I) \;\; 0 \le x_1, \; 0 \le x_2, \ldots \; 0 \le x_l$$

$$(II) \;\; 0 \le x_1 \le p_1, \; 0 \le x_2 \le p_1, \ldots \; 0 \le x_l \le p_1$$

E_n is called the *denumerant* (cf.Riordan).

Solution I.

$$\sum_{n=0}^{\infty} E_n t_n = (1 + t^{a_1} + t^2 a_1 + t^3 a_1 \ldots)(1 + t^{a_2} + t^2 a_2 + t^3 a_2 \ldots) \ldots (1 + t^{a_l} + t^2 a_l + t^3 a_l \ldots)$$

$$\sum_{n=0}^{\infty} E_n t_n = (1 - t^{a_1})^{-1} (1 - t^{a_2})^{-1} \ldots (1 - t^{a_l})^{-1} = \frac{P(t)}{Q(t)}.$$

Thus E_n is the coefficient of t^n in the *partial fraction expansion* of $\dfrac{P(t)}{Q(t)}$.

Example:

$$x + 2y = n$$

$$0 \leq x,\ 0 \leq y;\ a_1 = 1,\ a_2 = 2,\ a_3 = \ldots a_l = 0$$

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n t^n &= (1 + t^1 + t^2 + t^3 \ldots)(1 + t^2 + t^4 + t^6 \ldots) \\
&= \frac{1}{(1-t)(1-t^2)} \\
&= \frac{1}{(1-t)(1-t)(1+t)} \\
&= \frac{1}{(1-t)^2(1+t)} \\
&= \frac{1}{2(1-t)^2} + \frac{1}{4(1-t)} + \frac{1}{4(1+t)}. \\
\frac{1}{2(1-t)^2} &= \frac{1}{2}(1 + t^1 + t^2 + t^3 \ldots)^2 \\
&= \frac{1}{2}(1 + 2t^1 + 3t^2 + 4t^3 \ldots) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)t^n. \\
\frac{1}{4(1-t)} &= \frac{1}{4}(1 + t + t^2 + t^3 \ldots). \\
\frac{1}{4(1+t)} &= \frac{1}{4}(1 - t + t^2 - t^3 \ldots).
\end{aligned}$$

Upon collecting terms multiplying t^n we have for n even

$$E_n = \frac{n+1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{n+2}{2};$$

and for n odd

$$E_n = \frac{n+1}{2} + \frac{1}{4} - \frac{1}{4} = \frac{n+1}{2}.$$

Solution II.

Applying the matrix mechanics on convolution we have

$$\mathbf{A} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \dots \end{bmatrix}; \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ \dots \end{bmatrix}$$

Martix-vector multiplication yields the result:

$$\begin{bmatrix}
1 \\
1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 4 \\ 5 \\ 5 \\ 6 \\ 6 \\ 7 \\ 7 \\ 8 \\ 8 \\ 9 \\ 9 \\ 10 \\ 10 \\ 11 \end{bmatrix}.$$

6.3 Assignment 1.

- Combinatorial Analysis
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision January 2, 2015

Problems

I.1. *In how many ways can you change a dollar? (penny=1, nickel=5, dime=10, quarter=25, half-dollar=50)*

I. 2. *Let n stand for a non-negative integer and let A_n denote the number of solutions of the Diophantine equation*

$$x + 5y + 10z + 25u + 50v = n$$

in non-negative integers. Then the series

$$A_0 + A_1\zeta + A_2\zeta^2 + \ldots + A_n\zeta^n + \ldots$$

represents a rational function. Find it.

Proof: The first two questions will be solved simultaneously. There are x pieces of pennies, y pieces of nickels, z pieces of dimes etc. amounting to n cents. The generating function for the selection of pennies is

$$(1 + \zeta + \zeta^2 + \zeta^3 + \ldots)$$

by the *Rule of Sums* .Similarly, the generator for the selection of nickels is

$$(1 + \zeta^5 + \zeta^{10} + \zeta^{15} + \ldots)$$

and so forth for the other coins. By the *Rule of Products* the product of these functions represents $\Lambda(\zeta)$, the generator for the **change problem**

$$\Lambda(\zeta) = (1 + \zeta + \zeta^2 + \zeta^3 + \ldots) \times (1 + \zeta^5 + \zeta^{10} + \zeta^{15} + \ldots)\times$$

$$(1 + \zeta^{10} + \zeta^{20} + \zeta^{30} + \ldots) \times (1 + \zeta^{25} + \zeta^{50} + \zeta^{75} + \ldots) \times (1 + \zeta^{50} + \ldots).$$

Formal summation yields

$$(1+\zeta+\zeta^2+\zeta^3+\ldots)=\frac{1}{(1-\zeta)}$$

$$(1+\zeta^5+\zeta^{10}+\zeta^{15}+\ldots)=\frac{1}{(1-\zeta^5)},\ldots etc.$$

Therefore the generator

$$\Lambda(\zeta)=\frac{1}{(1-\zeta)}\frac{1}{(1-\zeta^5)}\frac{1}{(1-\zeta^{10})}\frac{1}{(1-\zeta^{25})}\frac{1}{(1-\zeta^{50})}$$

and can be expanded as

$$\Lambda(\zeta)=A_0+A_1\zeta+A_2\zeta^2+\ldots+A_n\zeta^n+\ldots$$

The required number is coefficient $A_{100}=292$.

Numerical Calculation:

D(n;1,2)

$$\sum_{i=0} q^i \sum_{j=0} (q^2)^j = \sum_{n=0} D(n;1,2) q^n$$

$$i + 2j = n$$

$$\begin{bmatrix}
1 \\
1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 4 \\ 5 \\ 5 \\ 6 \\ 6 \\ 7 \\ 7 \\ 8 \\ 8 \\ 9 \\ 9 \\ 10 \\ 10 \\ 11 \end{bmatrix}$$

D(n;1,2,5)

$$\sum_{i=0} q^i \sum_{j=0} (q^2)^j \sum_{k=0} (q^5)^k = \sum_{n=0} D(n;1,2,5) q^n$$

$$i + 2j + 5k = n; \; n = 0 \ldots 20$$

$$
\begin{bmatrix}
1 \\
0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 4 \\ 5 \\ 5 \\ 6 \\ 6 \\ 7 \\ 7 \\ 8 \\ 8 \\ 9 \\ 9 \\ 10 \\ 10 \\ 11 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 10 \\ 11 \\ 13 \\ 14 \\ 16 \\ 18 \\ 20 \\ 22 \\ 24 \\ 26 \\ 29 \end{bmatrix}
$$

D(n;1,2,5,10)

$$\sum_{i=0} q^i \sum_{j=0} (q^2)^j \sum_{k=0} (q^5)^k \sum_{l=0} (q^{10})^l = \sum_{n=0} D(n; 1, 2, 5, 10) q^n$$

$$i + 2j + 5k + 10l = n;\ \ n = 0 \ldots 20$$

$$
\begin{bmatrix}
1 \\
0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 10 \\ 11 \\ 13 \\ 14 \\ 16 \\ 18 \\ 20 \\ 22 \\ 24 \\ 26 \\ 29 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 11 \\ 12 \\ 15 \\ 16 \\ 19 \\ 22 \\ 25 \\ 28 \\ 31 \\ 34 \\ 40 \end{bmatrix}
$$

n	$D(n;1,2)$	$D(n;1,2,5)$	$D(n;1,2,5,10)$
0	1	1	1
1	1	1	1
2	2	2	2
3	2	2	2
4	3	3	3
5	3	4	4
6	4	5	5
7	4	6	6
8	5	7	7
9	5	8	8
10	6	10	11
11	6	11	12
12	7	13	15
13	7	14	16
14	8	16	19
15	8	18	22
16	9	20	25
17	9	22	28
18	10	24	31
19	10	26	34
20	11	29	40

$$\sum_{n=0}^{n=20} D(n;1,2,5,10) = 292$$

$$100 = \overbrace{(n \cdot 5)}^{5,10,25,50} + \underbrace{(100 - n \cdot 5)}_{1}$$

I. 3. *In how many ways can you put the necessary stamps in one row on a letter using* $2, 4, 6, 8$ *cent stamps? The postage is* 10 *cents. Different arrangements of the same value are regarded as different ways.*

I. 4. *We call* B_n *the number of all possible sums with value* n *(*n *positive integer) whose terms are* $1, 2, 3$ *or* 4*. (Two sums consisting of the same terms but in different order are regarded as different.) Then the series*

$$1 + B_1\zeta + B_2\zeta^2 + \ldots + B_n\zeta^n + \ldots$$

represents a rational function of ζ. Which one?

Proof: Problems 3 and 4 solved together.

First, consider the **stamp problem** .

$$10 = 2+4+4 = 4+4+2 = 4+2+4 = 6+2+2 = 8+2$$

are all examples of *partitions* of number 10 with the restriction that the *parts* are specified $\{2, 4, 6, 8\}$ and different arrangements of the same value are regarded as different ways. How many different partitions are there?

Second, the number of possible sums with value n and with the given restrictions is a generalization of the **stamp** problem. For the first part we have four choices

$$\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4$$

by the *Rule of Sums* . For two parts

$$(\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)^2 = \zeta^2 + 2\zeta^3 + 3\zeta^4 + 4\zeta^5 + 3\zeta^6 + 2\zeta^7 + \zeta^8$$

For s parts the generator is

$$(\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)^s.$$

Therefore summation for $1, 2, \ldots s \ldots$ gives

$$1 + \sum_{n=1}^{n=\infty} B_n = 1 + (\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4) + (\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)^2 +$$

$$\ldots + +(\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)^s + \ldots = \frac{1}{1 - (\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)}.$$

The required number is $B_5 = 15$.

Numerical Calculation: First we show that the recursion

$$B_n = B_{n-1} + B_{n-2} + B_{n-3} + B_{n-4}$$

follows from the definition. The set of possible sums with value n can be divided into four disjoint subsets. The number of possible sums with value

n and with the restrictions that the last term is k equals B_{n-k}, $k = 1, 2, 3, 4$. Every sum with value n belongs to exactly one of the above subsets.

$$(\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)^1 = \zeta^1 + \zeta^2 + \zeta^3 + \zeta^4$$

$$(\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)^2 = \zeta^2 + 2\zeta^3 + 3\zeta^4 + 4\zeta^5 + 3\zeta^6 + \dots$$

$$(\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)^3 = \zeta^3 + 3\zeta^4 + 6\zeta^5 + 10\zeta^6 + \dots$$

$$(\zeta^1 + \zeta^2 + \zeta^3 + \zeta^4)^4 = \zeta^4 + \dots$$

$$B_1 = 1;\ B_2 = 2;\ B_3 = 4;\ B_4 = 8;\ B_5 = 15.$$

I. 5. *Someone owns a set of eight weights of* $\{1, 1, 2, 5, 10, 10, 20, 50\}$ *grams, respectively. In how many different ways can* 78 *grams be composed of such weights? (Replacing one weight by another of the same value counts as different way.)*

I. 6. *In how many different ways can one weigh* 78 *grams if the weighs may be placed in both* pans of the scales and the same weights are used as in problem **5** ?

I. 7. We consider the sum of the form

$$\epsilon_1 + \epsilon_2 + 2\epsilon_3 + 5\epsilon_4 + 10\epsilon_5 + 10\epsilon_6 + 20\epsilon_7 + 50\epsilon_8$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \dots \epsilon_8$ assume the values of 0 or 1. We call C_n the number of different sums with value n. Write the polynomial

$$C_0 + C_1\zeta + C_2\zeta^2 + \dots + C_{99}\zeta^{99}$$

as a product.

I. 8. Let $\epsilon_1, \epsilon_2, \epsilon_3, \dots \epsilon_8$ assume the values of -1,0,1. Modify problem **7** accordingly. Let D_n denote the number of different sums of value n. Find the factorization of the following expression (function of ζ)

$$\sum_{n=-99}^{n=99} D_n\zeta^n.$$

Solutions: Problems **5,6,7,8** form a *proportion* : **5** is to **7** as **6** is to **8** .
First weighing problem 5,7 , all weighs in one pan

$$\sum_{n=0}^{n=99} C_n\zeta^n = (1+\zeta)^2(1+\zeta^2)(1+\zeta^5)(1+\zeta^{10})^2(1+\zeta^{20})(1+\zeta^{50}).$$

Second weighing problem 6,8 , weights may be placed on both pans

$$\sum_{n=-99}^{n=99} D_n\zeta^n = (\zeta^{-1}+1+\zeta)^2(\zeta^{-2}+1+\zeta^2)(\zeta^{-5}+1+\zeta^5)\times$$

$$(\zeta^{-10}+1+\zeta^{10})^2(\zeta^{-20}1+\zeta^{20})(\zeta^{-50}1+\zeta^{50}).$$

The required numbers are

$$C_{78} = 4, \quad D_{78} = 20.$$

The calculations of A_n, B_n, C_n, D_n can be carried out easily on programable calculators.

I. 9. *These protoype problems can be generalized as follows:*

Solutions: Change problem

$$a_1x_1 + a_2x_2 + \ldots + a_lx_l = n,$$

$$\Lambda(\zeta) = \frac{1}{(1-\zeta^{a_1})}\frac{1}{(1-\zeta^{a_2})}\cdots\frac{1}{(1-\zeta^{a_l})} = \sum_{n=0}^{n=\infty} A_n\zeta^n.$$

Stamp problem

$$\Lambda(\zeta) = \frac{1}{1-(\zeta^{a_1}+\zeta^{a_2}+\ldots+\zeta^{a_l})} = \sum_{n=0}^{n=\infty} B_n\zeta^n.$$

First weighing problem , all weights in one pan

$$\Lambda(\zeta) = (1+\zeta^{a_1})(1+\zeta^{a_2})(\ldots)(1+\zeta^{a_l}) = \sum_{n=0}^{n=\infty} C_n\zeta^n.$$

Second weighing problem , weights may be placed on both pans

$$\Lambda(\zeta) = (\zeta^{-a_1}+1+\zeta^{a_1})(\zeta^{-a_2}+1+\zeta^{a_2})(\ldots)(\zeta^{-a_l}+1+\zeta^{a_l}) = \sum_{n=0}^{n=\infty} D_n\zeta^n$$

Closing Remarks i) Partitions of number n can be demonstrated by *Ferrers graphs* . For example $10 = 4 + 4 + 2$ in the **stamp** problem becomes

$$\bullet\ \bullet\ \bullet\bullet$$

$$\bullet\ \bullet\ \bullet\bullet$$

$$\bullet\ \bullet$$

or, if read vertically, $10 = 3 + 3 + 2 + 2$.
ii) On programable calculators the coefficients A_n, B_n, C_n, D_n are calculated by a simple subroutine that multiplies a polynomial by another polynomial.

$$p(x) = a_0x^0 + a_1x^1 + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$$

$$q(x) = b_0x^0 + b_1x^1 + b_2x^2 + \ldots + b_{m-1}x^{m-1} + b_mx^m$$

$$r(x) = r_0x^0 + r_1x^1 + r_2x^2 + \ldots + r_{m+n-1}x^{m+n-1} + r_{m+n}x^{m+n}$$

$$p(x)q(x) = r(x)$$

$$A(i) = (a_0, a_1, a_2, \ldots, a_{n-1}, a_n)$$

$$B(j) = (b_0, b_1, b_2, \ldots, b_{m-1}, b_m)$$

$$C(k) = (c_0, c_1, c_2, \ldots, c_{m+n-1}, c_{m+n})$$

Input

$$A, B, n, m$$

Algorithm

$$C \leftarrow 0$$

for $i = 0, 1, 2, 3, \ldots n$ **do**
for $j = 0, 1, 2, 3, \ldots m$ **do**

$$k = i + j$$

$$C(k) = C(k) + A(i) * B(j)$$

end for (j)
end for (i)
Output

$$C$$

6.4 Assignment 2.

Summary

- Combinatorial Analysis
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision January 2, 2015

Problems

I.10. *An assembly of p persons elects a committee consisting of n of its members. How many different committees can they choose?*

Solution:

$$\binom{p}{n} = \frac{p(p-1)(p-2)\dots(p-n+1)}{n!} = \frac{p!}{(p-n)!n!}.$$

It is also possible to interpret this problem by means of *generators :* Given a person say *Melvin* ; he is either selected ξ^1 or not selected ξ^0, a simple alternative. Therefore the generator for Melvin is $(1+\xi)$.There are p persons, p independent alternatives,

$$(1+\xi)^p = 1 + \binom{p}{1}\xi^1 + \binom{p}{2}\xi^2 \dots + \binom{p}{n}\xi^n \dots + \xi^p.$$

The is the *binomial theorem* , of course, and the coefficients,in this order, give the number of ways to select $1, 2, 3, \dots, n, \dots p$ persons out of p persons.

I 11. *There are p persons sharing n dollars. In how many ways can they distribute the money?*

Solution: One person, say *Melvin* can get $0, 1, 2, \dots, r, \dots$ dollars. The generator for Melvin is

$$(1 + \xi^1 + \xi^2 + \dots + \xi^r + \dots) = \frac{1}{1-\xi}$$

after formal summation. There are p persons, therefore the generator is

$$(1+\xi^1+\xi^2+\ldots+\xi^r+\ldots)^p = \frac{1}{(1-\xi)^p} = (1-\xi)^{-p}.$$

By **Lemma 1.**

$$(1-\xi)^{-p} = \sum_0^\infty \left(\begin{array}{c} p+r-1 \\ p-1 \end{array} \right) \xi^r$$

therefore the required number is $r=n$

$$\left(\begin{array}{c} p+n-1 \\ p-1 \end{array} \right).$$

I 12. *There are p persons sharing n dollars, each getting at least one dollar. In how many different ways can they do it?*

Solution: Again, using *Melvin* as an example, he can get $1, 2, \ldots, r, \ldots$ dollars. So the generator for Melvin is

$$(\xi^1+\xi^2+\ldots+\xi^r+\ldots) = \frac{\xi}{1-\xi}$$

after formal summation. The generator for p persons is

$$(\xi^1+\xi^2+\ldots+\xi^r+\ldots)^p = \frac{1}{(1-\xi)^p} = \xi^p(1-\xi)^{-p}$$

By **Lemma 1.** Then by **Lemma 2.**

$$(\xi^1+\xi^2+\ldots+\xi^r+\ldots)^p = \sum_{r=p}^\infty \left(\begin{array}{c} r-1 \\ p-1 \end{array} \right) \xi^r.$$

Therefore the required number is $r=n$

$$\left(\begin{array}{c} n-1 \\ p-1 \end{array} \right).$$

I 13. *Consider the general homogeneous polynomial of degree n in p variables $x_1, x_2, \ldots x_p$. How many terms does it have?*

Discussion: Let

$$S_i = 1 + x_i + x_i^2 + x_i^3 + \ldots, \; i = 1, 2, \ldots p$$

be infinite (formal) sums. Write

$$G(x_1, x_2, \ldots x_p) = \Pi_{i=1}^{i=p} S_i = \Pi_{i=1}^{i=p}(1 + x_i + x_i^2 + x_i^3 + \ldots)$$

for the product of these sums. Termwise multiplication takes one and only one element from each sum S_i; $x_1^{k_1}$ from S_1; $x_2^{k_2}$ from S_2; and so on. The typical *term* $x_1^{k_1} x_2^{k_2} \ldots x_p^{k_p}$ has a combined degree of $k_1 + k_2 + \ldots + k_p$. Collect the terms that have the same combined degree r, into a homogeneous polynomial of degree r in p variables $x_1, x_2, \ldots x_p$, call it f_r

$$f_r(x_1, x_2, \ldots x_p) = \sum_{k_1 + k_2 \ldots + k_p = r} x_1^{k_1} x_2^{k_2} \ldots x_p^{k_p}; \; 0 \leq k_i; \; i = 1, 2, \ldots p,$$

in f_r each term is multiplied by coefficient 1. Note further that f_r has as many terms as the general homogeneous polynomial of degree n in p variables. Next, sum f_r over $r = 0, 1, 2, \ldots$

$$G(x_1, x_2, \ldots x_p) = \sum_{r=0}^{r=\infty} f_r = \sum_{r=0}^{r=\infty} x_1^{k_1} x_2^{k_2} \ldots x_p^{k_p}$$

where

$$k_1 + k_2 + \ldots + k_p = r; \; 0 \leq k_i; \; i = 1, 2, \ldots p$$

in the last sum. Thus

$$G(x_1, x_2, \ldots x_p) = \Pi_{i=1}^{i=p}(1 + x_i + x_i^2 + x_i^3 + \ldots) = \sum_{r=0}^{r=\infty} x_1^{k_1} x_2^{k_2} \ldots x_p^{k_p}.$$

Substitution

$$x_1 = x_2 = \ldots = x_p = \xi$$

gives

$$G(\xi, \xi, \ldots, \xi) = (1 + \xi + \xi^2 + \xi^3 + \ldots)^p = \sum_{r=p}^{\infty} \binom{p + r - 1}{p - 1} \xi^r.$$

Thus the generator for this problem is identical to the one in **I 11** .

Solution: The general homogeneous polynomial of degree n in p variables $x_1, x_2, \ldots x_p$ has

$$\binom{p+n-1}{p-1}$$

terms.

Example:

$$p = 3$$

$$S_i = 1 + x_i + x_i^2 + x_i^3 + \ldots, \; i = 1, 2, 3$$

$$G(x_1, x_2, x_3) = \sum_{r=0}^{r=\infty} x_1^{k_1} x_2^{k_2} x_3^{k_3} =$$

$$1 + (x_1 + x_2 + x_3)_1 + (x_1x_1 + x_1x_2 + x_1x_3 + x_2x_2 + x_2x_3 + x_3x_3)_2$$

$$(x_1^3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_1x_2x_3 + x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3)_3 + \ldots$$

$$|(\ldots)_0| = 1 = \binom{3+0-1}{3-1} = \binom{2}{2} \surd$$

$$|(\ldots)_1| = 3 = \binom{3+1-1}{3-1} = \binom{3}{2} \surd$$

$$|(\ldots)_2| = 6 = \binom{3+2-1}{3-1} = \binom{4}{2} \surd$$

$$|(\ldots)_3| = 10 = \binom{3+3-1}{3-1} = \binom{5}{2} \surd$$

etc.

See also **Lemma 3** : eight objects partitioned into six cells

$$\bullet| \bullet \bullet || \bullet \bullet \bullet || \bullet \bullet$$

$$x_1|x_2x_2||x_4x_4x_4||x_6x_6 = x_1x_2^2x_4^3x_6^2.$$

I 14. *Any positive integer admits a unique representation in the binary system* $(1, 2, 4, 8, 16, \ldots)$.

Discussion This is a well-known result from computer science and number theory. Let us express the first 7 positive integers

$$1 = 1; 2 = 2; 3 = 2 + 1; 4 = 4; 5 = 4 + 1; 6 = 4 + 2; 7 = 4 + 2 + 1$$

by the first 3 numbers of the binary system $\{1, 2, 4\}$. Working with generators $(1+\xi^1)$,$(1+\xi^2)$, and $(1+\xi^4)$ as we did in solving the protoype problems, we can compose these integers and zero

$$(1 + \xi^1)(1 + \xi^2)(1 + \xi^4) = \xi^0 + \xi^1 + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7$$

in a unique way. Take another step:

$$\begin{aligned} (1 + \xi^1)(1 + \xi^2)(1 + \xi^4)(1 + \xi^8) &= \\ (1 + \xi^1 + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7)(1 + \xi^8) &= \\ \xi^0 + \xi^1 + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7 &+ \\ \xi^{0+8} + \xi^{1+8} + \xi^{2+8} + \xi^{3+8} + \xi^{4+8} + \xi^{5+8} + \xi^{6+8} + \xi^{7+8} &= \\ (\xi^0 + \ldots + \xi^7) + (\xi^8 + \ldots + \xi^{15}) &= \sum_{i=0}^{i=15} \xi^i. \end{aligned}$$

This is a *(formal) power series* where the indeterminate has *exponents* $\{0, 1, 2, \ldots 15\}$ and *uniform coefficients* 1. Thus the first four generators yield zero and the positive integers that are less than $16 = 2^4$. Observe how multiplication by $(1 + \xi^8)$ preserves and shifts the power series to higher exponents.

Next, we state our induction hypothesis :

$$\Pi_{i=0}^{i=n}(1 + \xi^{2^i}) = \sum_{i=0}^{i=2^{n+1}-1} \xi^i$$

for any positive integer n. This statement is true for $n = 2, 3$. The general case follows by induction on n.

Solution: For any positive integer $n > 2$ the integers $\{0, 1, 2, \ldots (2^{n+1}-1)\}$ can be represented in the binary system $(1, 2, 4, 8, 16, \ldots, 2^n)$ in A_i different ways

$$\Pi_{i=0}^{i=n}(1 + \xi^{2^i}) = \sum_{i=0}^{i=2^{n+1}-1} A_i\xi^i.$$

Since

$$A_i = 1, \forall i,$$

there is a unique representation for each positive integer.

Solution:

$$\begin{aligned}(1+\xi)(1+\xi^2)(1+\xi^4)(1+\xi^8)\ldots &= \frac{(1-\xi^2)}{(1-\xi)}\frac{(1-\xi^4)}{(1-\xi^2)}\frac{(1-\xi^8)}{(1-\xi^4)}\frac{(1-\xi^{16})}{(1-\xi^8)}\ldots \\ \frac{1}{1-\xi} &= \xi^0+\xi^1+\xi^2+\xi^3+\xi^4+\xi^5+\ldots\end{aligned}$$

using

$$(1+\xi^{2k}) = (1-\xi^k)(1+\xi^k)$$

and formal summation. See also prototype weighing problems.

I 15. *Any positive integer admits a unique representation in the ternary system with + or − signs,* $(\pm 1, \pm 3, \pm 9, \pm 27, \pm 81, \ldots)$.

Solution: Again, see prototype weighing problems: using both pans of the scale. *Quasipolynomials* and summation of (short) *geometric series* :

$$(\xi^{-1}+1+\xi^1) = \xi^{-1}(1+\xi^1+\xi^2) = \xi^{-1}\frac{\xi^3-1}{\xi-1}$$

$$(\xi^{-3}+1+\xi^3) = \xi^{-3}(1+\xi^3+\xi^6) = \xi^{-3}\frac{\xi^9-1}{\xi^3-1}\ldots$$

$$(\xi^{-3^n}+1+\xi^{3^n}) = \xi^{-3^n}(1+\xi^{3^n}+\xi^{2*3^n}) = \xi^{-3n}\frac{\xi^{3^{n+1}}-1}{\xi^{3^n}-1}$$

The generator is the product of the quasipolynomials

$$\Pi_n(\xi^{-3^n}+1+\xi^{3^n}) = \xi^{-1}\frac{\xi^3-1}{\xi-1}\xi^{-3}\frac{\xi^9-1}{\xi^3-1}\ldots\xi^{-3^n}\frac{\xi^{3^{n+1}}-1}{\xi^{3^n}-1} =$$

$$(\xi^{-1}\xi^{-3}\ldots\xi^{-3^n})\left(\frac{\xi^3-1}{\xi-1}\right)\left(\frac{\xi^9-1}{\xi^3-1}\right)\ldots\left(\frac{\xi^{3^{n+1}}-1}{\xi-1}\right) =$$

$$(\xi^{-1}\xi^{-3}\ldots\xi^{-3^n})\left(\frac{\xi^{3^{n+1}}-1}{\xi^{3^n}-1}\right)=\xi^{-(1+3+\ldots+3^n)}\left(\frac{\xi^{3^{n+1}}-1}{\xi-1}\right).$$

$$(1+3+\ldots+3^n)=\frac{3^{n+1}-1}{3-1}=\frac{3^{n+1}-1}{2}=N$$

$$\left(\frac{\xi^{3^{n+1}}-1}{\xi-1}\right)=\left(\frac{\xi^{2N+1}-1}{\xi-1}\right)=1+\xi+\xi^2+\ldots+\xi^{2N}$$

$$\Pi_n(\xi^{-3^n}+1+\xi^{3^n})=\xi^{-N}(1+\xi+\xi^2+\ldots+\xi^{2N})=\xi^{-N}+\xi^{-N+1}+\ldots+\xi^{N-1}+\xi^N.$$

The generators produce a quasipolynomial with uniform coefficients equal to 1. Thus the number of ways an integer $\{-N,-N+1,\ldots N-1,N\}$ can be represented in the signed ternary sytem is egual to $\{A_{-N},A_{-N+1},\ldots,A_{N-1},A_N\}$, respectively.

$$\Pi_{i=0}^{i=n}(\xi^{-3^n}+1+\xi^{3^n})=A_{-N}\xi^{-N}+A_{-N+1}\xi^{-N+1}+\ldots+A_{N-1}\xi^{N-1}+A_N\xi^N;$$

$$N=\frac{3^{n+1}-1}{2}.$$

All A-s are equal to 1.

Example:

$$\{\pm1,\pm3,\pm9\};\ n=2;\ N=\frac{\xi^{3^{2+1}}-1}{2}=13.$$

$$(\xi^{-1}+1+\xi^1)=\xi^{-1}\frac{\xi^3-1}{\xi-1}$$

$$(\xi^{-3}+1+\xi^3)=\xi^{-3}\frac{\xi^9-1}{\xi^3-1}$$

$$(\xi^{-3^2}+1+\xi^{3^2})=\xi^{-3^2}\frac{\xi^{3^{2+1}}-1}{\xi^{3^2}-1}$$

$$(\xi^{-1}+1+\xi^1)(\xi^{-3}+1+\xi^3)(\xi^{-3^2}+1+\xi^{3^2})=\xi^{-1}\frac{\xi^3-1}{\xi-1}\xi^{-3}\frac{\xi^9-1}{\xi^3-1}\xi^{-3^2}\frac{\xi^{3^{2+1}}-1}{\xi^{3^2}-1}=$$

$$(\xi^{-13})\frac{\xi^{27}-1}{\xi-1} = (\xi^{-13})(\xi^{26}+\xi^{25}+\ldots+\xi^{1}+1) =$$

$$(\xi^{-13}+\xi^{-12}+\ldots+\xi^{0}+\ldots+\xi^{11}+\xi^{12}+\xi^{13}).$$

-13	= -9-3-1	13	= 9+3+1
-12	= -9-3	12	= 9+3
-11	= -9-3+1	11	= 9+3-1
-10	= -9-1	10	= 9+1
-9	= -9	9	= 9
-8	= -9+1	8	= 9-1
-7	= -9+3-1	7	= 9-3+1
-6	= -9+3	6	= 9-3
-5	= -9+3+1	5	= 9-3-1
-4	= -3-1	4	= 3+1
-3	= -3	3	= 3
-2	= -3+1	2	= 3-1
-1	= -1	1	= 1

I 16. *Write*

$$(1+q\zeta)(1+q\zeta^2)(1+q\zeta^4)(1+q\zeta^8)(1+q\zeta^{16})\ldots = a_0+a_1\zeta+a_2\zeta^2+a_3\zeta^3+\ldots.$$

Find the general formula for a_n.

Solution: Recall **I 14.**

$$S_0 = (1+\xi)(1+\xi^2)(1+\xi^4)(1+\xi^8)\ldots = \xi^0+\xi^1+\xi^2+\xi^3+\xi^4+\xi^5+\ldots$$

Then

$$S_1 = (1+q\zeta)(1+q\zeta^2)(1+q\zeta^4)(1+q\zeta^8)\ldots =$$

$$1+q\zeta+q\zeta^2+q^2\zeta^3+q\zeta^4+q^2\zeta^5+q^2\zeta^6+q^3\zeta^7+q\zeta^8\ldots$$

and after replacing q by $q_1, q_2, q_4\ldots$, respectively

$$S_2 = (1+q_1\zeta^1)(1+q_2\zeta^2)(1+q_4\zeta^4)(1+q_8\zeta^8)\ldots =$$

$$1+q_1\zeta+q_2\zeta^2+q_1q_2\zeta^3+q_4\zeta^4+q_1q_4\zeta^5+q_2q_4\zeta^6+q_1q_2q_4\zeta^7+q_8\zeta^8\ldots$$

Compare this to the *unique binary representation* of numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$

$$1 \equiv 1;\ 2 \equiv 10;\ 3 \equiv 11;\ 4 \equiv 100;$$

$$5 \equiv 101;\ 6 \equiv 110;\ 7 \equiv 111;\ 8 \equiv 1000.$$

Thus series S_0 proves that each positive integer has a unique representation in the binary system, series S_1 shows how many 1-s each representation has, and S_2 details which powers of 2 are included in the composition. Therefore

$$a_n = q^{F(n)}$$

where $F(n)$ is the number of digits 1 in the unique binary representation of positiv integer n.

I 17. *Consider the expansion*

$$(1-a)(1-b)(1-c)(1-d)\ldots = 1 - a - b + ab - c + ac + bc - abc - d + \ldots$$

What is the sign of the $n-th$ *term?*

Solution: **Deception!** Consider S_2:

$$S_2 = (1 + q_1\zeta^1)(1 + q_2\zeta^2)(1 + q_4\zeta^4)(1 + q_8\zeta^8)\ldots =$$

$$1 + q_1\zeta + q_2\zeta^2 + q_1q_2\zeta^3 + q_4\zeta^4 + q_1q_4\zeta^5 + q_2q_4\zeta^6 + q_1q_2q_4\zeta^7 + q_8\zeta^8\ldots$$

First replace the linearly ordered finite set $\{a, b, c, d \ldots z\}$ by the linearly ordered set $\{q_1\zeta^1, q_2\zeta^2, q_4\zeta^4 \ldots q_{2^k}\zeta^{2^k}\}$

$$S_3 = (1 - q_1\zeta^1)(1 - q_2\zeta^2)(1 - q_4\zeta^4)(1 - q_8\zeta^8)\ldots =$$

$$1 - q_1\zeta - q_2\zeta^2 + q_1q_2\zeta^3 - q_4\zeta^4 + q_1q_4\zeta^5 + q_2q_4\zeta^6 - q_1q_2q_4\zeta^7 + q_8\zeta^8\ldots$$

Notice the patterns of $+$s and $-$s are the same. Next, make

$$q = q_1 = q_2 = q_4 = \ldots$$

which reduces

$$(1-a)(1-b)(1-c)(1-d)\ldots = 1 - a - b + ab - c + ac + bc - abc - d + \ldots =$$

to

$$1 - q\zeta^1 - q\zeta^2 + q^2\zeta^3 - q\zeta^4 + q^2\zeta^5 + q^2\zeta^6 - q^3\zeta^7 - q\zeta^8 \ldots$$

the two series have the same pattern of siqns. Finally, set

$$\zeta = 1$$

$$1 - q\zeta^1 - q\zeta^2 + q^2\zeta^3 - q\zeta^4 + q^2\zeta^5 + q^2\zeta^6 - q^3\zeta^7 - q\zeta^8 \ldots$$

and the sign of the n-th term is $(-1)^{F(n)}$.

6.5 Assignment 3.

- Combinatorial Analysis
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision January 2, 2015

Problems

I.18. Prove the identity

$$\begin{array}{rl} (1+\xi+\xi^2+\ldots+\xi^9) & \times \\ (1+\xi^{10}+\xi^{20}+\ldots+\xi^{90}) & \times \\ (1+\xi^{100}+\xi^{200}+\ldots+\xi^{900} & \times \\ \ldots & = \dfrac{1}{1-\xi} \end{array}$$

Solution *Repeated application of summation formula of finite geometric series*

$$1+\xi+\xi^2+\ldots+\xi^{n-1} = \frac{\xi^n-1}{\xi-1}$$

because

$$(1+\xi+\xi^2+\ldots+\xi^{n-1})(\xi-1) = \xi^n-1$$

$$1+\xi+\xi^2+\ldots+\xi^9 = \frac{\xi^{10}-1}{\xi-1}$$

$$1+\xi^{10}+\xi^{20}+\ldots+\xi^{90} = \frac{\xi^{100}-1}{\xi^{10}-1}$$

$$1+\xi^{100}+\xi^{200}+\ldots+\xi^{900} = \frac{\xi^{1000}-1}{\xi^{100}-1}$$

$$\frac{\xi^{10}-1}{\xi-1}\times\frac{\xi^{100}-1}{\xi^{10}-1}\times\frac{\xi^{1000}-1}{\xi^{100}-1}\ldots = \frac{-1}{\xi-1} = \frac{1}{1-\xi}$$

I.18.2 *In a legistlative assembly there are $2n+1$ seats and three parties. In how many different ways can the seats be distributed among the parties so that no party attains a majority against the coalition of the other two?*

Solution Obviously, majority is $n+1$ seats, therefore the restriction means that the truncated series

$$1+\xi+\xi^2+\ldots+\xi^n$$

is the generator for a party. Since there are three parties the generator for combinations with limited repetitions of objects (seats) of 3 kinds with restriction that each kind may appear no more than n times is:

$$(1+\xi+\xi^2+\ldots+\xi^n)^3 = \sum_{k=0} E_k\xi^k.$$

The number of ways is E_{2n+1}. Our plan is to proceed from combinations with unlimited repetitions to combinations with limited repetitions. Two facts are required :

i) The generator for combinations with unlimited repetitions of objects of p kinds and no restriction on the number of times each kind appears is:

$$\begin{aligned}(1+\xi+\xi^2+\ldots+\xi^n+\ldots)^p &= (1-\xi)^{-p} \\ &= \sum_{k=0}^{\infty}\binom{p+k-1}{k}\xi^k \\ &= \sum_{k=0}^{\infty}\binom{p+k-1}{p-1}\xi^k\end{aligned}$$

For proof see Lemma 1.

ii)

$$(1+\xi+\xi^2+\ldots+\xi^n)^2 = 1+2\xi+3\xi^2+\ldots+(n+1)\xi^n+\ldots$$

Next, set $p=3$ and write

$$A=(1+\xi+\xi^2+\ldots+\xi^n),\ B=(\xi^{n+1}+\ldots).$$

$$(A+B)^3 = A^3+3A^2B+AB^2+B^3$$

$$(A+B)^3 = \ldots + \binom{2n+3}{2}\xi^{2n+1} + \ldots$$

by **i)** . Further,

$$A^3 = \ldots + a_{2n+1}\xi^{2n+1} + \ldots$$

$$\begin{aligned}
3A^2B &= 3(1+\xi+\xi^2+\ldots+\xi^n)^2(\xi^{n+1}+\ldots) \\
&= 3(1+2\xi+3\xi^2+\ldots+(n+1)\xi^n+\ldots)(\xi^{n+1}+\xi^{n+2}+\ldots) \\
&= \ldots + 3(1+2+3+\ldots+(n+1))\xi^{2n+1} + \ldots \\
&= \ldots b_{2n+1}\xi^{2n+1} + \ldots
\end{aligned}$$

$$\binom{2n+3}{2}\xi^{2n+1} = (a_{2n+1}+b_{2n+1})\xi^{2n+1}$$

because (AB^2+B^3) does not contribute to the coefficient of ξ^{2n+1}.

$$b_{2n+1} = 3(1+2+3+\ldots+(n+1)) = 3\frac{(n+2)(n+1)}{2} = 3\binom{n+2}{2}$$

$$\begin{aligned}
a_{2n+1} &= \binom{2n+3}{2} - 3\binom{n+2}{2} \\
&= \frac{(2n+3)(2n+12)}{2} - \frac{(n+2)(n+1)}{2} \\
&= \frac{(n+1)[2(2n+3)-3(n+2)]}{2} \\
&= \frac{(n+1)n}{2} \\
&= \frac{(n+1)n}{2} \\
&= \binom{n+1}{2}.
\end{aligned}$$

I.19.

$$(1+\xi^1)(1+\xi^2)(1+\xi^3)(1+\xi^4)\ldots = \frac{1}{(1-\xi)(1-\xi^3)(1-\xi^5)(1-\xi^7)\ldots}$$

Solution Write

$$\prod_{n=1}(1+\xi^n) = \prod_{n=1}(1-\xi^{2n-1})^{-1}.$$

$$\{1,2,3,\ldots n\ldots\} = \cup_{n=1}\{(2n-1), 2*(2n-1), 2^2*(2n-1), 2^3*(2n-1),\ldots\}$$

This is a decomposition of positive integers into disjoint sets.Let us list some of the disjoint sets in question:

$$S_2 = \{1,2,4,8,16,32,\ldots\}$$

$$S_3 = \{3,6,12,24,48,\ldots\}$$

$$S_5 = \{5,10,20,40,80,\ldots\}$$

$$S_7 = \{7,14,28,56,112,\ldots\}$$

$$S_9 = \{9,18,36,72,144,\ldots\}$$

$$S_{11} = \{11,22,44,88,\ldots\}$$

Note that the first set contains the powers of 2. The second, and subsequent sets are multiples of odd numbers.Every positive integer P belongs to exactly one set. Write

$$P = 2^a p^b q^c \ldots; \quad p, q, \ldots \text{primes}.$$

If $a = 0$ then

$$P = p^b q^c \ldots = 2n-1, \quad \text{for some} \quad n$$

$$\exists : \{P, 2*P, 2^2*P, 2^3*P,\ldots\}.$$

If $a > 0$ then

$$p^b q^c \ldots = 2m-1, \quad \text{for some} \quad m; \quad P = 2^a(2m-1)$$

$$\exists : \{(2m-1),\ldots,2^a(2m-1),\ldots\}.$$

Multiplication takes place set by set.

$$S_2: \quad (1+\xi^1)(1+\xi^2)(1+\xi^4)(1+\xi^8)\ldots = (1-\xi^1)^{-1}$$

because [cf.**I.14.**]

$$\begin{aligned}(1+\xi^1) &= \frac{(1-\xi^2)}{(1-\xi)}\\(1+\xi^2) &= \frac{(1-\xi^4)}{(1-\xi^2)}\\(1+\xi^4) &= \frac{(1-\xi^8)}{(1-\xi^4)}\\(1+\xi^8) &= \frac{(1-\xi^{16})}{(1-\xi^8)}\\&\vdots\end{aligned}$$

$$(1+\xi^1)(1+\xi^2)(1+\xi^4)(1+\xi^8)\dots$$

$$= \frac{(1-\xi^2)}{(1-\xi)}\frac{(1-\xi^4)}{(1-\xi^2)}\frac{(1-\xi^8)}{(1-\xi^4)}\frac{(1-\xi^{16})}{(1-\xi^8)}$$

Upon cancelling $(1-\xi^2)$,$(1-\xi^4)$, etc. and using $|\xi|<1$

$$(1+\xi^1)(1+\xi^2)(1+\xi^4)(1+\xi^8)\dots = \frac{1}{(1-\xi)}$$

This method works with $\alpha = 3, 5, \dots (2n-1), \dots$ as well.

$$S_\alpha: \quad (1+\xi^\alpha)(1+\xi^{2\alpha})(1+\xi^{4\alpha})(1+\xi^{8\alpha})\dots = (1-\xi^\alpha)^{-1}$$

$$\begin{aligned}(1+\xi^{1\alpha}) &= \frac{(1-\xi^{2\alpha})}{(1-\xi^\alpha)}\\(1+\xi^{2\alpha}) &= \frac{(1-\xi^{4\alpha})}{(1-\xi^{2\alpha})}\\(1+\xi^{4\alpha}) &= \frac{(1-\xi^{8\alpha})}{(1-\xi^{4\alpha})}\\(1+\xi^{8\alpha}) &= \frac{(1-\xi^{16\alpha})}{(1-\xi^{8\alpha})}\\&\vdots\end{aligned}$$

$$(1+\xi^{1\alpha})(1+\xi^{2\alpha})(1+\xi^{4\alpha})(1+\xi^{8\alpha})\dots$$

$$= \frac{(1-\xi^{2\alpha})}{(1-\xi)} \frac{(1-\xi^{4\alpha})}{(1-\xi^{2\alpha})} \frac{(1-\xi^{8\alpha})}{(1-\xi^{4\alpha})} \frac{(1-\xi^{16\alpha})}{(1-\xi^{8\alpha})} \cdots$$

etc.

$$(1+\xi^{1\alpha})(1+\xi^{2\alpha})(1+\xi^{4\alpha})(1+\xi^{8\alpha})\ldots = (1-\xi^{\alpha})^{-1}$$

as claimed. Then the final result comes after multiplying the contributions of S_2, $S_\alpha, \alpha = 3, 5, 7, 9, \ldots$

$$(1-\xi^1)^{-1}(1-\xi^3)^{-1}(1-\xi^5)^{-1}\ldots(1-\xi^\alpha)^{-1}\ldots = \prod_{n=1}(1-\xi^{2n-1})^{-1}.$$

I.20 *Each positive integer can be decomposed into a sum of different positive integers in as many ways as it can be decomposed into a sum of equal or different odd positive integers.*

Solution Consider the decompositions of 6 into sums with different terms

$$6 = 1+5 = 2+4 = 1+2+3,$$

and with odd terms

$$1+5 = 3+3 = 1+1+1+3 = 1+1+1+1+1+1.$$

The generator for decomposing a number into different (unequal) terms is

$$(1+\xi)(1+\xi^2)(1+\xi^3)+\ldots) = 1 + \sum_{k=0} E_k \xi^k$$

The coefficient E_n is equal to the number of ways how a combination (selection) of different (unequal) numbers

$$\{1, 2, 3, \ldots\}$$

can add up to n. Each number can be used no more than once, order is unimportant. As shown in **I.19.**:

$$(1+\xi^1)(1+\xi^2)(1+\xi^3)(1+\xi^4)\ldots = \frac{1}{(1-\xi)(1-\xi^3)(1-\xi^5)(1-\xi^7)\ldots}$$

The generator for decomposing a number into odd numbers is

$$\begin{aligned} & (1+\xi+\xi^2+\xi^3\ldots) \text{ multiples of } 1 \\ \times\ & (1+\xi^3+\xi^6+\xi^9\ldots) \text{ multiples of } 3\ldots \\ \times\ & (1+\xi^\alpha+\xi^{2\alpha}+\xi^{3\alpha}\ldots) \text{ multiples of } \alpha \end{aligned}$$

where $\alpha > 3$ runs through the odd numbers. Summing the series one by one

$$\begin{aligned} (1+\xi+\xi^2+\xi^3\ldots) &= (1-\xi)^{-1} \\ (1+\xi^3+\xi^6+\xi^9\ldots) &= (1-\xi^3)^{-1} \\ (1+\xi^\alpha+\xi^{2\alpha}+\xi^{3\alpha}\ldots) &= (1-\xi^\alpha)^{-1} \end{aligned}$$

and collecting the results

$$(1-\xi)^{-1}(1-\xi^3)^{-1}\ldots(1-\xi^\alpha)^{-1}\ldots = \frac{1}{(1-\xi)(1-\xi^3)(1-\xi^5)(1-\xi^7)\ldots}$$

completes the proof.

I.21 *It is possible to write the positive integer n in $2^{n-1}-1$ ways as a sum of smaller integers. Two sums are that differ in the order of terms only are now regarded as different.*

Solution 1.: Suppose there are n balls in a row:

$$\bullet\bullet\bullet\ldots\bullet\bullet$$

Then there are $n-1$ options to insert one separator into any of the $n-1$ gaps, for example

$$\bullet\bullet|\bullet\ldots\bullet\bullet$$

which models the partition $2+(n-2)=n$. There are $(n-1)(n-2)$ options to insert two separators, for example

$$\bullet|\bullet\bullet|\ldots\bullet\bullet$$

which is associated with $1+2+(n-3)=n$. Any gap between two balls can have no more than one separator. Oviously,

$$\bullet|\bullet|\bullet|\ldots|\bullet|\bullet$$

is associated with $1+1+1+\ldots+1=n$.

$$\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-1}=2^{n-1}-1.$$

Solution 1.: ”Stamp” problem with minor changes. Let B_n be the number of all possible sums with value n whose terms are $1, 2, 3, \ldots, n$ (Conditions are relaxed to include n.) The generators for sums with one, two, etc. k terms are

$$(\xi + \xi^2 + \ldots + \xi^n),\ (\xi + \xi^2 + \ldots + \xi^n)^2, \ldots (\xi + \xi^2 + \ldots + \xi^n)^k, \ldots,$$

respectively.

Next, we sum the generators and expand the result

$$1 + \sum_{k=1}^{\infty}(\xi + \xi^2 + \ldots + \xi^n)^k = 1 + \sum_{k=1}^{\infty} B'_k \xi^k$$

The number of all possible sums with value n

$$1 + \sum_{k=1}^{\infty}(\xi + \xi^2 + \ldots + \xi^n + \ldots)^k$$

I 22. *The total number of non-negative integral solutions of the following Diophantine equations is $n+1$:*

$$x + 2y = n;\ 2x + 3y = n - 1;\ 3x + 4y = n - 2;\ \ldots$$

$$\ldots\ nx + (n+1)y = 1;\ (n+1)x + (n+2)y = 0.$$

Solution :

6.6 Assignment 4.

- Combinatorial Analysis
- *Hua, Riordan*
- Last revision January 2, 2015

Notes on partition Let q be real or complex, $|q| < 1$ and let us define the following functions

$$\begin{aligned} q_0 &= \prod_{n=1}^{\infty}(1-q^{2n}) \\ q_1 &= \prod_{n=1}^{\infty}(1+q^{2n}) \\ q_2 &= \prod_{n=1}^{\infty}(1+q^{2n-1}) \\ q_3 &= \prod_{n=1}^{\infty}(1-q^{2n-1}). \end{aligned}$$

Recall that the infinite product

$$(1+u_1)(1+u_2)(1+u_3)\ldots = \prod_{k=1}^{\infty}(1+u_k)$$

converges to $P \neq 0$ if $\lim P_n = P$ where

$$P_n = (1+u_1)(1+u_2)(1+u_3)\ldots(1+u_n), \quad u_k \neq -1, \forall k.$$

A necessary and sufficient condition condition that $\prod(1+u_k)$ converge absolutely is that $\sum u_k$ converge absolutely.

Proposition 1. If $|q| < 1$, then

$$q_1 q_2 q_3 = 1.$$

Proof (1): Consider index sets $\{n\}, \{2n-1\}, \{2n\}$:

$$\{2n\} \cup \{2n-1\} = \{n\}, \;\{2n\} \cap \{2n-1\} = \{\emptyset\}$$

$$q_0 q_3 = \prod_{n=1}^{\infty}(1-q^{2n}) \prod_{n=1}^{\infty}(1-q^{2n-1}) = \prod_{n=1}^{\infty}(1-q^{n}).$$

$$q_1 q_2 = \prod_{n=1}^{\infty}(1+q^{2n}) \prod_{n=1}^{\infty}(1+q^{2n-1}) = \prod_{n=1}^{\infty}(1+q^{n})$$

$$q_0 q_1 q_2 q_3 = \prod_{n=1}^{\infty}(1-q^{n}) \prod_{n=1}^{\infty}(1+q^{n}) = \prod_{n=1}^{\infty}(1-q^{2n}) = q_0$$

$$q_1 q_2 q_3 = 1.$$

Proof (2):

$$\begin{aligned} q_2 q_3 &= \left(\prod_{n=1}^{\infty}(1+q^{2n-1})\right)\left(\prod_{n=1}^{\infty}(1-q^{2n-1})\right) \\ &= \prod_{n=1}^{\infty}\Big((1+q^{2n-1})(1-q^{2n-1})\Big) \\ &= \prod_{n=1}^{\infty}(1-q^{2(2n-1)}) \end{aligned}$$

$$\begin{aligned} q_1 &= \prod_{n=1}^{\infty}(1+q^{2n}) \\ &= \prod_{n=1}^{\infty}(1+q^{2(2n-1)}) \prod_{n=1}^{\infty}(1+q^{4(2n-1)}) \prod_{n=1}^{\infty}(1+q^{8(2n-1)}) \dots \end{aligned}$$

because

$$\{2n\} = \{2(2n-1)\} \cup \{4(2n-1)\} \cup \{8(2n-1)\} \dots$$

To verify this decomposition let M be an even number, $M \epsilon \{2n\}$. Then

$$M = 2^{\alpha} p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$$

where $\alpha > 0$ and $p_i, i = 1, \dots k$ are odd primes. This representation is unique. Thus M belongs to one and only one subset, $\{2^\alpha(2n-1)\}$ and $p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k} = 2m - 1$ for some integer m. On the other hand, there is no odd number in any of the $\{2^\alpha(2n-1)\}$ subsets. Therefore the decomposition is valid. Here is a numerical example of "taking out the powers of 2" from even numbers up to 40:

$$\{2n\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \dots 40, \dots\}$$

$$\{2(2n-1)\} = \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, \dots\}$$

$$\{4(2n-1)\} = \{4, 12, 20, 28, 36, 44, \dots\}$$

$$\{8(2n-1)\} = \{8, 24, 40, 48, \dots\}$$

$$\{16(2n-1)\} = \{16, 48, \dots\}$$

$$\{32(2n-1)\} = \{32, 96, \dots\}$$

$$\{64(2n-1)\} = \{64, \dots\}$$

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \dots 40\} =$$

$$\{2, 6, 10, 14, 18, 22, 26, 30, 34, 38\} \cup \{4, 12, 20, 28, 36\} \cup \{8, 24, 40\} \cup \{16\} \cup \{32\}.$$

End of numerical example.

$$\begin{aligned} q_1 q_2 q_3 &= \prod_{n=1}^{\infty}(1 - q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \dots \\ &= \left(\prod_{n=1}^{\infty}(1 - q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{2(2n-1)})\right) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)}) \dots \\ &= \left(\prod_{n=1}^{\infty}(1 - q^{4(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)})\right) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \dots \\ &= \left(\prod_{n=1}^{\infty}(1 - q^{8(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)})\right) \prod_{n=1}^{\infty}(1 + q^{16(2n-1)}) \dots = 1. \end{aligned}$$

The lowest exponent of q is rising yet the infinite products have the same limit.Why? Write

$$K(q) = \prod_{n=1}^{\infty}(1 - q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \dots$$

$$K(q^2) = \prod_{n=1}^{\infty}(1 - q^{4(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)} \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \ldots$$

$$K(q^4) = \prod_{n=1}^{\infty}(1 - q^{8(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{16(2n-1)}) \ldots$$

$$K(q) = K(q^2) = K(q^4) = \ldots = K(0) = 1$$

$K(\ )$ is invariant under the substitution of q^2 for q, $\lim q^{2n} = 0$.

6.7 Assignment 5.

- Combinatorial Analysis
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision January 2, 2015

Problems

I 22. *The total number of non-negative integral solutions of the following Diophantine equations is* $n+1$*:*

$$x+2y=n;\ 2x+3y=n-1;\ 3x+4y=n-2;\ \dots$$

$$\dots\ nx+(n+1)y=1;\ (n+1)x+(n+2)y=0.$$

Solution : Consider the first Diophantine equation:

$$x+2y=n.$$

The generators for the solution are

$$1+\xi+\xi^2+\xi^3+\dots$$

and

$$1+\xi^2+\xi^4+\xi^6+\dots$$

where the exponents of ξ are multiples of 1 and 2 , respectively. The number of non-negative integral solutions is the coefficient of ξ^n in the product of

$$(1+\xi+\xi^2+\xi^3+\dots)\times(1+\xi^2+\xi^4+\xi^6+\dots).$$

where the series are extended past n. Only the first $n+1$ terms of the first generator and not more than the first $[\frac{n}{2}]+1$ terms of the second generator can contribute to the coefficient of ξ^n in the product. After formal summation

$$(1+\xi+\xi^2+\xi^3+\dots)\times(1+\xi^2+\xi^4+\xi^6+\dots)=\frac{1}{(1-\xi)(1-\xi^2)}.$$

Therefore the number of non-negative integral solutions of the Diophantine equation $x + 2y = n$ is the coefficient of ξ^n in the expansion of

$$\frac{1}{(1-\xi)(1-\xi^2)}.$$

Similarly, the number of the non-negative integral solutions of the Diophantine equation $2x + 3y = n - 1$ is the coefficient of ξ^{n-1} in the product of

$$(1 + \xi^2 + \xi^4 + \xi^6 + \ldots) \times (1 + \xi^3 + \xi^6 + \xi^9 + \ldots).$$

Note that the exponents of ξ are multiples of 2 and 3 , respectively. Again, after formal summation

$$(1 + \xi^2 + \xi^4 + \xi^6 + \ldots) \times (1 + \xi^3 + \xi^6 + \xi^9 + \ldots) = \frac{1}{(1-\xi^2)(1-\xi^3)}.$$

Combining the first two Diophantine equations $x+2y = n$ and $2x+3y = n-1$, the total number of their non-negative integral solutions is the coefficient of ξ^n in the expansion of

$$\frac{1}{(1-\xi)(1-\xi^2)} + \frac{\xi}{(1-\xi^2)(1-\xi^3)}.$$

This idea can be extended to include the remaining equations

$$3x + 4y = n - 2; \ldots nx + (n + 1)y = 1;\ (n + 1)x + (n + 2)y = 0.$$

Write

$$S(\xi) = \frac{1}{(1-\xi)(1-\xi^2)} + \frac{\xi}{(1-\xi^2)(1-\xi^3)} + \ldots + \frac{\xi^\nu}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})} + \ldots$$

The total number of non-negative integral solutions of the Diophantine equations

$$x + 2y = n;\ 2x + 3y = n - 1;\ 3x + 4y = n - 2;\ \ldots$$

$$\ldots\ nx + (n + 1)y = 1;\ (n + 1)x + (n + 2)y = 0.$$

is the coefficient of ξ^n in the expansion of $S(x)$.

Next, observe the partial fraction decomposition

$$\frac{\xi^\nu}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})} = \frac{1}{\xi(1-\xi)}\left(\frac{1}{1-\xi^{\nu+1}} - \frac{1}{1-\xi^{\nu+2}}\right).$$

Because

$$\frac{1}{\xi(1-\xi)}\left(\frac{1}{1-\xi^{\nu+1}}-\frac{1}{1-\xi^{\nu+2}}\right)=$$

$$\frac{1}{\xi(1-\xi)}\left(\frac{1-\xi^{\nu+2}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}-\frac{1-\xi^{\nu+1}}{(1-\xi^{\nu+2})(1-\xi^{\nu+1})}\right)=$$

$$\frac{1}{\xi(1-\xi)}\left(\frac{(1-\xi^{\nu+2})-(1-\xi^{\nu+1})}{(1-\xi^{\nu+2})(1-\xi^{\nu+1})}\right)=$$

$$\frac{1}{\xi(1-\xi)}\left(\frac{(\xi^{\nu+1}-\xi^{\nu+2})}{(1-\xi^{\nu+2})(1-\xi^{\nu+1})}\right)=\frac{1}{\xi(1-\xi)}\left(\frac{\xi^{\nu}(\xi^{1}-\xi^{2})}{(1-\xi^{\nu+2})(1-\xi^{\nu+1})}\right)=$$

$$\frac{1}{\xi(1-\xi)}\left(\frac{\xi^{\nu}(\xi^{1}(1-\xi))}{(1-\xi^{\nu+2})(1-\xi^{\nu+1})}\right)=\frac{\xi^{\nu}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}.$$

Therefore

$$S(\xi)=\frac{1}{\xi(1-\xi)}\sum_{\nu=0}^{\nu=\infty}\left(\frac{1}{1-\xi^{\nu+1}}-\frac{1}{1-\xi^{\nu+2}}\right).$$

This is a telescoping sum, after cancellation

$$S(\xi)=\frac{1}{\xi(1-\xi)}\left(\frac{1}{1-\xi}-1\right)=$$

$$(\xi^{-1})(1+\xi+\xi^2+\xi^3+\ldots)\times(\xi+\xi^2+\xi^3+\ldots)=$$

$$(1+\xi+\xi^2+\xi^3+\ldots)\times(1+\xi+\xi^2+\xi^3+\ldots)=$$

$$(1+\xi+\xi^2+\xi^3+\ldots)^2=1+2\xi+3\xi^2+4\xi^3+\ldots+(n+1)\xi^n+\ldots$$

as claimed.

I 23. The total number N of non-negative integral solutions of the following Diophantine equations

$$x+2y=n-1;\ 2x+3y=n-3;\ 3x+4y=n-5;\ \ldots$$

is smaller than $n+2$; moreover the difference $n+2-N$ is equal to the number of divisors of $n+2$.

Remark: The total number of divisors of number n (including 1 and n) is denoted by $d(n)$. So if $n = p^a q^b \dots$ with $p, q \dots$ distinct primes then

$$d(n) = (a+1)(b+1)\dots$$

Discussion: The left-hand sides of the Diophantine equations are the same as in **I. 22.** Skipping the discussion on generators and formal summations we have

Claim I: the total number N of non-negative integral solutions is the coefficient of ξ^{n-1} in the expansion of

$$S(\xi) = \sum_{\nu=0}^{\nu=\infty} \frac{\xi^{2\nu}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}.$$

Proof of Claim I: The number of solutions to the first Diophantine equation is equal to A_{n-1} in the expansion

$$\frac{1}{(1-\xi^1)(1-\xi^2)} = \sum_{\nu=0}^{\nu=\infty} A_\nu \xi^\nu.$$

Similarly, the numbers of solutions to the second and third Diophantine equations are given by B_{n-3} in the expansion

$$\frac{1}{(1-\xi^2)(1-\xi^3)} = \sum_{\nu=0}^{\nu=\infty} B_\nu \xi^\nu$$

and C_{n-5}

$$\frac{1}{(1-\xi^3)(1-\xi^4)} = \sum_{\nu=0}^{\nu=\infty} C_\nu \xi^\nu,$$

respectively. Therefore the combined number of the non-negative integral solutions to the first three equations

$$x + 2y = n-1;\ 2x + 3y = n-3; 3x + 4y = n-5;$$

is $A_{n-1} + B_{n-3} + C_{n-5}$.

Multiplication by ξ^2 and ξ^4 shifts B_ν to $B_{\nu+2}$ and C_ν to $C_{\nu+4}$, respectively, and the required number is the coefficient of ξ^{n-1} in the final expansion, whenever $n \geq 5$

Write

$$\frac{1}{(1-\xi^1)(1-\xi^2)} + \frac{\xi^2}{(1-\xi^2)(1-\xi^3)} + \frac{\xi^4}{(1-\xi^3)(1-\xi^4)} =$$

$$\sum_{\nu=0}^{\nu=\infty} A_\nu \xi^\nu + \xi^2 \sum_{\nu=0}^{\nu=\infty} B_\nu \xi^\nu + \xi^4 \sum_{\nu=0}^{\nu=\infty} C_\nu \xi^\nu =$$

$$\sum_{\nu=0}^{\nu=\infty} A_\nu \xi^\nu + \sum_{\nu=0}^{\nu=\infty} B_\nu \xi^{\nu+2} + \sum_{\nu=0}^{\nu=\infty} C_\nu \xi^{\nu+4} = \sum_{\nu=0}^{\nu=\infty} (A_\nu + B_{\nu-2} + C_{\nu-4})\xi^\nu,$$

$$B_{-2} = B_{-1} = C_{-4} = C_{-3} = C_{-2} = C_{-1} = 0.$$

Therefore, in this manner, we can build a solution that includes the non-negative integral solutions to all Diophantine equations :

$$S(\xi) = \frac{1}{(1-\xi^1)(1-\xi^2)} + \frac{\xi^2}{(1-\xi^2)(1-\xi^3)} + \frac{\xi^4}{(1-\xi^3)(1-\xi^4)} + \dots$$

Claim II:

$$S(\xi) = \sum_{\nu=0}^{\nu=\infty} \frac{\xi^{2\nu}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})} = \frac{1}{1-\xi} \sum_{\nu=0}^{\nu=\infty} \xi^{\nu-1} \left(\frac{1}{1-\xi^{\nu+1}} - \frac{1}{1-\xi^{\nu+2}} \right).$$

Check:

$$\frac{\xi^{\nu-1}}{1-\xi} \left(\frac{1}{1-\xi^{\nu+1}} - \frac{1}{1-\xi^{\nu+2}} \right) =$$

$$\frac{\xi^{\nu-1}}{1-\xi} \left(\frac{1-\xi^{\nu+2}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})} - \frac{1-\xi^{\nu+1}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})} \right) =$$

$$\frac{\xi^{\nu-1}}{1-\xi} \left(\frac{\xi^{\nu+1}-\xi^{\nu+2}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})} \right) = \frac{\xi^{\nu-1}}{1-\xi} \left(\frac{\xi^{\nu+1}(1-\xi)}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})} \right) =$$

$$\frac{\xi^{2\nu}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}.\surd$$

Claim III:

$$\begin{aligned}
S(\xi) &= \frac{1}{1-\xi}\sum_{\nu=0}^{\nu=\infty}\xi^{\nu-1}\left(\frac{1}{1-\xi^{\nu+1}}-\frac{1}{1-\xi^{\nu+2}}\right)\\
&= \frac{1}{1-\xi}\left[\xi^{-1}\left(\frac{1}{1-\xi^{1}}-\frac{1}{1-\xi^{2}}\right)\right]_{(\nu=0)}\\
&+ \frac{1}{1-\xi}\left[\xi^{0}\left(\frac{1}{1-\xi^{2}}-\frac{1}{1-\xi^{3}}\right)\right]_{(\nu=1)}\\
&+ \frac{1}{1-\xi}\left[\xi^{1}\left(\frac{1}{1-\xi^{3}}-\frac{1}{1-\xi^{4}}\right)\right]_{(\nu=2)}\\
&+ \frac{1}{1-\xi}\left[\xi^{2}\left(\frac{1}{1-\xi^{4}}-\frac{1}{1-\xi^{5}}\right)\right]_{(\nu=3)} + \dots\\
&+ \frac{1}{1-\xi}\left[\xi^{n-1}\left(\frac{1}{1-\xi^{n+1}}-\frac{1}{1-\xi^{n+2}}\right)\right]_{(\nu=n)}\\
&+ \frac{1}{1-\xi}\left[\xi^{n}\left(\frac{1}{1-\xi^{n+2}}-\frac{1}{1-\xi^{n+3}}\right)\right]_{(\nu=n+1)} + \dots
\end{aligned}$$

Next combine the terms of

$$\frac{1}{1-\xi^{\nu+2}}$$

from two consecutive terms $\nu, \nu+1$, here found along slanting diagonals:

$$\begin{aligned}
S(\xi) &= \frac{1}{\xi}\frac{1}{(1-\xi)^2}\\
&+ \frac{1}{1-\xi}\left(\xi^{0}\frac{1}{1-\xi^{2}}-\xi^{-1}\frac{1}{1-\xi^{2}}\right)_{(\nu=0,1)}\\
&+ \frac{1}{1-\xi}\left(\xi^{1}\frac{1}{1-\xi^{3}}-\xi^{0}\frac{1}{1-\xi^{3}}\right)_{(\nu=1,2)}\\
&+ \frac{1}{1-\xi}\left(\xi^{2}\frac{1}{1-\xi^{4}}-\xi^{1}\frac{1}{1-\xi^{4}}\right)_{(\nu=2,3)} + \dots\\
&+ \frac{1}{1-\xi}\left(\xi^{n}\frac{1}{1-\xi^{n+2}}-\xi^{n-1}\frac{1}{1-\xi^{n+2}}\right)_{(\nu=n,n+1)} + \dots
\end{aligned}$$

$$
\begin{aligned}
S(\xi) &= \frac{1}{\xi}\frac{1}{(1-\xi)^2} \\
&+ \frac{(\xi^0-\xi^{-1})}{1-\xi}\left(\frac{1}{1-\xi^2}\right) + \frac{(\xi^1-\xi^0)}{1-\xi}\left(\frac{1}{1-\xi^3}\right) \\
&+ \frac{(\xi^2-\xi^1)}{1-\xi}\left(\frac{1}{1-\xi^4}\right) + \ldots + \frac{(\xi^n-\xi^{n-1})}{1-\xi}\left(\frac{1}{1-\xi^{n+2}}\right)\ldots
\end{aligned}
$$

Noting that

$$\left(\frac{1}{\xi^3}\frac{\xi}{1-\xi} - \frac{1}{\xi^3}\frac{\xi}{1-\xi}\right) = 0$$

and

$$\left(\frac{\xi^3}{\xi^3}\right) = 1$$

we have

$$
\begin{aligned}
S(\xi) &= \frac{1}{\xi}\frac{1}{(1-\xi)^2} + \left(\frac{1}{\xi^3}\frac{\xi}{1-\xi} - \frac{1}{\xi^3}\frac{\xi}{1-\xi}\right) \\
&+ \left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^0-\xi^{-1})}{1-\xi}\left(\frac{1}{1-\xi^2}\right) + \left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^1-\xi^0)}{1-\xi}\left(\frac{1}{1-\xi^3}\right) \\
&+ \left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^2-\xi^1)}{1-\xi}\left(\frac{1}{1-\xi^4}\right) + \ldots + \left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^n-\xi^{n-1})}{1-\xi}\left(\frac{1}{1-\xi^{n+2}}\right)\ldots
\end{aligned}
$$

By examining $S(\xi)$ term by term we obtain
First:

$$
\begin{aligned}
\frac{1}{\xi}\frac{1}{(1-\xi)^2} + \frac{1}{\xi^3}\left(\frac{\xi}{1-\xi}\right) &= \frac{1}{\xi^2}\frac{\xi}{(1-\xi)^2} + \frac{1}{\xi^2}\left(\frac{1}{1-\xi}\right) = \\
\frac{1}{\xi^2}\left(\frac{\xi}{(1-\xi)^2} + \left(\frac{1}{1-\xi}\right)\right) &= \frac{1}{\xi^2}\left(\frac{\xi}{(1-\xi)^2} + \frac{1-\xi}{(1-\xi)^2}\right) = \frac{1}{\xi^2}\frac{1}{(1-\xi)^2}.
\end{aligned}
$$

Second:

$$-\frac{1}{\xi^3}\frac{\xi}{1-\xi}$$

Third:

$$
\begin{aligned}
\left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^0-\xi^{-1})}{1-\xi}\left(\frac{1}{1-\xi^2}\right) &= \left(\frac{\xi}{\xi^3}\right)\frac{(\xi^0-\xi^{-1})}{1-\xi}\left(\frac{\xi^2}{1-\xi^2}\right) = \\
\left(\frac{1}{\xi^3}\right)\frac{(\xi-1)}{1-\xi}\left(\frac{1}{1-\xi^2}\right) &= -\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^2}{1-\xi^2}\right).
\end{aligned}
$$

Fourth:

$$\left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^1-\xi^0)}{1-\xi}\left(\frac{1}{1-\xi^3}\right) \;=\; -\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^3}{1-\xi^3}\right)\ldots$$

N-th:

$$\begin{aligned}\left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^{n-3}-\xi^{n-4})}{1-\xi}\left(\frac{1}{1-\xi^{n-1}}\right) \;&=\; \left(\frac{1}{\xi^3}\right)\frac{\xi-1}{1-\xi}\left(\frac{\xi^{n-1}}{1-\xi^{n-1}}\right)= \\ &-\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^{n-1}}{1-\xi^{n-1}}\right).\end{aligned}$$

$$\begin{aligned} S(\xi) &= \frac{1}{\xi^2}\frac{1}{(1-\xi)^2} \\ &- \left(\frac{1}{\xi^3}\right)\left(\frac{\xi}{1-\xi}\right)-\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^2}{1-\xi^2}\right) \\ &- \left(\frac{1}{\xi^3}\right)\left(\frac{\xi^3}{1-\xi^3}\right)\ldots-\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^{n-1}}{1-\xi^{n-1}}\right)\ldots \\ &= \frac{1}{\xi^2}\frac{1}{(1-\xi)^2}-\left(\frac{1}{\xi^3}\right)\sum_{\nu=1}^{\nu=\infty}\frac{\xi^\nu}{1-\xi^\nu}. \end{aligned}$$

Next, we examine the main term

$$\begin{aligned}\frac{1}{\xi^2}\frac{1}{(1-\xi)^2} &= \frac{1}{\xi^2}*\frac{1}{(1-\xi)}*\frac{1}{(1-\xi)} \\ &= \frac{1}{\xi^2}*(1+\xi+\xi^2+\ldots)*(1+\xi+\xi^2+\ldots) \\ &= \frac{1}{\xi^2}*(1+2\xi+3\xi^2+\ldots+(n+1)\xi^n+(n+2)\xi^{n+1}+(n+3)\xi^{n+2}) \\ &= (\ldots+(n+1)\xi^{n-2}+(n+2)\xi^{n-1}+(n+3)\xi^n+\ldots)\end{aligned}$$

This shows that the coefficient of ξ^{n-1} is $(n+2)$ in the main term. Now we come to the secondary term without $\left(\frac{1}{\xi^3}\right)$.

Claim IV:

$$\sum_{\nu=1}^{\nu=\infty}\frac{\xi^\nu}{1-\xi^\nu} \;=\; \tau(1)\xi+\tau(2)\xi^2+\ldots+\tau(\nu)\xi^\nu+$$

where $\tau(\nu)$ denotes the number of divisors of ν.

Discussion of Claim IV: Let us pick a number - say 6 - and let us see what contributes to the coefficient of ξ^6.

$$\begin{aligned}
\frac{\xi^1}{1-\xi^1} &= \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \underline{\xi^6} + \xi^7 + \dots \\
\frac{\xi^2}{1-\xi^2} &= \xi^2 + \xi^4 + \underline{\xi^6} + \xi^8 + \dots \\
\frac{\xi^3}{1-\xi^3} &= \xi^3 + \underline{\xi^6} + \xi^9 + \dots \\
\frac{\xi^6}{1-\xi^6} &= \underline{\xi^6} + \xi^{12} \dots
\end{aligned}$$

Clearly, expansions for $\nu = 4, 5$, or $\nu \geq 7$ would not contribute. Therefore the coefficient of ξ^6 is equal to 4, the number of ν -s that divide 6. Further, pick another number - say 7 - . Only two expansions have ξ^7 in them:

$$\begin{aligned}
\frac{\xi^1}{1-\xi^1} &= \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \underline{\xi^7} + \xi^8 + \dots \\
\frac{\xi^7}{1-\xi^7} &= \underline{\xi^7} + \xi^{14} + \dots
\end{aligned}$$

Of course, 7 is a prime, and $\tau(7) = 2$. This not a proof, but we are convinced.

6.8 Miscellaneous Notes

6.8.1 Current interests

Spartan Old School Tutorials: 3 levels, undergraduate standards

Computer Skills: Basic programming with Fortran and C; Math Tools, Graphics, Numerical and Symbolic Computations; Latex typesetting, On-line Tutorials.

Classics in Pure Math: *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis, Konvexer Körper , Vinogradov-Turán*

Classics in Applied Math : *Ciarlet, Birkhoff-Rota, Geÿza Freud*

Mathematical Modelling and Numerical Analysis: *Smith, Morton- Mayers, Ames*

6.8.2 Envoy

” I am afraid, Watson, that I shall have to go. ”
said Holmes as we sat down together to our breakfast on one morning.
” Go! Where to?”
” To Dartmoor, to King’s Pyland.”
I was not surprised.