Pinter Consulting
New Series Nos. 15.

J K Pinter, Dr.Tech.

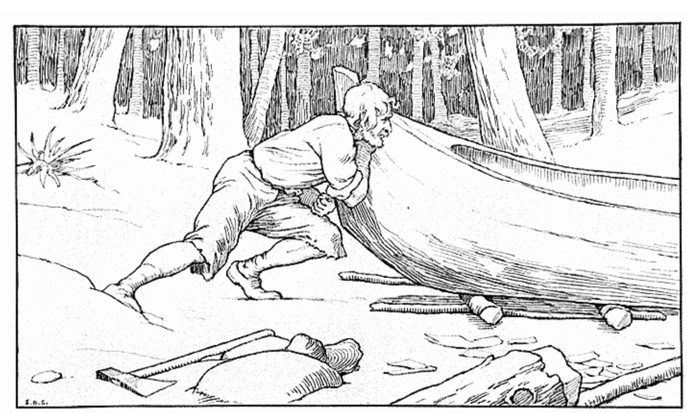
September 30, 2017

Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

- **Private studies for professional development**
- **béláim, gondolkozzunk, amig lehet**
- Socratic Programme for 2017
- Pólya - Szegö : Aufgaben und Lehrsätze aus der Analysis
- Memoirs of Applied Mathematics
- Spartan Old School Tutorials
- Én ős Robinson - lelkemet
  Szent álomfrász lelkemet
  Paskolják kis Büdösök
  Paskolják kis Büdösök.
- Vah, nyavalya*, nyavalya
  Veh, büdösök, nyavalya
  De Robinson Krausz nem beszél
  De Robinson Krausz nem beszél.

  nyavalya* = köszvény



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Introduction

Pinter Consulting of Calgary, Alberta practices Mathematics, promotes clear thinking and offers Consultations, Tutorials and Seminars in Mathematics.

Summary

Work in progress.

Contents

15.0 Assignment 41.

Summary

- Mathematical Modelling
- Separation of variables, Fourier series
- Last revision September 30, 2017

Heat equation, model problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \quad x\varepsilon[0,1], \quad t \geq 0, \tag{15.1}$$

$$u(0,t) = u(1,t) = 0, \tag{15.2}$$

$$u(x,0) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} \leq x \leq 1. \end{cases} \tag{15.3}$$

Separation of variables

Assume

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{\partial u}{\partial t} = X(x)T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X"(x)T(t)$$

Therefore

$$X(x)T'(t) = X"(x)T(t)$$

or

$$\frac{T'(t)}{T(t)} = \frac{X"(x)}{X(x)}.$$

This implies

$$\frac{T'(t)}{T(t)} = \frac{X"(x)}{X(x)} = -\lambda^2,$$

and we have two separate ordinary differential equations.

The first one is for $X(x)$:

$$-X"(x) = \lambda^2 X(x) \tag{15.4}$$

$$X(0) = X(1) = 0 \tag{15.5}$$

It is a Sturm-Liouville problem with eigenvalues

$$\lambda_n^2 = (n\pi)^2$$

and corresponding eigenfunctions

$$X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots$$

The second one is

$$T'(t) = -\lambda^2 T(t).$$

A solution is

$$T_n(t) = \ exp(-\lambda_n^2 t) = \ exp(-(n\pi)^2 t) \ \ t \geq 0, \quad n = 1, 2, \ldots$$

(The choice of $-\lambda^2$ is justified because the solution does not grow out of bounds.) Therefore we claim that each

$$u_n(x,t) = X_n(x)T_n(t) = \sin(n\pi x)\ exp(-(n\pi)^2 t) \tag{15.6}$$

satisfies heat equation (15.1), and boundary conditions (15.2):

$$\frac{\partial u_n(x,t)}{\partial t} = -(n\pi)^2 \sin(n\pi x)\ exp(-(n\pi)^2 t)$$

$$\frac{\partial^2 u_n(x,t)}{\partial x^2} = -(n\pi)^2 \sin(n\pi x)\ exp(-(n\pi)^2 t)$$

$$u_n(0,t) = \sin(n\pi 0)\ exp(-(n\pi)^2 t) = 0$$

$$u_n(1,t) = \sin(n\pi)\ exp(-(n\pi)^2 t) = 0.$$

Superposition, Fourier series

Next, we develop the Fourier series approximation of the "hat function" which is the initial condition for the model problem.

(i) The function $u^0(x)$ is defined on $[0,1]$ by

$$u^0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \tag{15.7}$$

Show that

$$u^0(x) = \sum_{m=1}^{\infty} a_m \sin(m\pi x)$$

where

$$a_m = (8/m^2\pi^2)\sin(\frac{1}{2}m\pi).$$

(Morton - Mayers)

Proof: Let us define

$$F_0(x) = \begin{cases} u^0(x) & 0 \le x \le 1, \\ -u^0(-x) & -1 \le x \le 0. \end{cases} \tag{15.8}$$

$F_0(x)$ is an odd 2-periodic extension of $u^0(x)$ in $(-1,1)$.

$$0 = 2\int_0^1 u^0(x)\cos(m\pi x)dx.$$

$$a_m = 2\int_0^1 u^0(x)\sin(m\pi x)dx.$$

Thus a_m is the coefficient in the Fourier sine series expansion of $u^0(x)$. Evaluation by integration on $[0,\frac{1}{2}]$ and $[\frac{1}{2},0]$

$$a_m = 2\int_0^{\frac{1}{2}} u^0(x)\sin(m\pi x)dx + 2\int_{\frac{1}{2}}^1 u^0(x)\sin(m\pi x)dx$$

using the following lemma:

$$\int z\sin(az)dz = \frac{\sin(az)}{a^2} - \frac{z\cos(az)}{a} + C.$$

First integral:

$$\begin{aligned} 2\int_0^{\frac{1}{2}} 2x\sin(m\pi x)dx &= 4\int_0^{\frac{1}{2}} x\sin(m\pi x)dx \\ &= 4\left[\frac{\sin(m\pi x)}{m^2\pi^2} - \frac{x\cos(m\pi x)}{m\pi}\right]_0^{\frac{1}{2}} \\ &= 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi}. \end{aligned}$$

Second integral:

$$\begin{aligned} 2\int_{\frac{1}{2}}^1 2\sin(m\pi x)dx &= 4\left[\frac{-1}{m\pi}\cos(m\pi x)\right]_{\frac{1}{2}}^1 \\ &= \frac{-4}{m\pi}\cos(m\pi 1) + \frac{4}{m\pi}\cos(m\pi\frac{1}{2}) \end{aligned}$$

Third integral:

$$\begin{aligned} 2\int_{\frac{1}{2}}^1 -2x\sin(m\pi x)dx &= -4\int_{\frac{1}{2}}^1 x\sin(m\pi x)dx \\ &= -4\left[\frac{\sin(m\pi x)}{m^2\pi^2} - \frac{x\cos(m\pi x)}{m\pi}\right]_{\frac{1}{2}}^1 \\ &= -4\frac{\sin(m\pi 1)}{m^2\pi^2} + \frac{4\cos(m\pi 1)}{m\pi} \\ &+ 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi}. \end{aligned}$$

From the first and third integral

$$4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} + 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} = 8\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2}.$$

From the first, second and third

$$-2\frac{\cos(m\pi\frac{1}{2})}{m\pi} + \frac{4}{m\pi}\cos(m\pi\frac{1}{2}) - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi} = 0$$

From the second and third

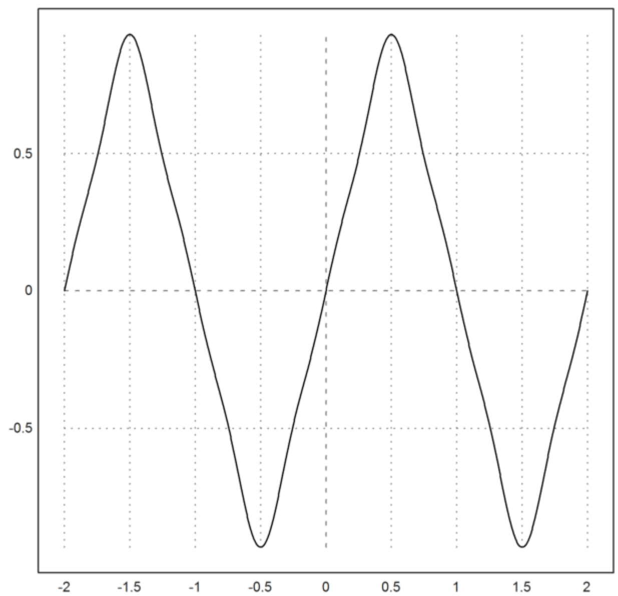
$$-\frac{4}{m\pi}\cos(m\pi 1) + \frac{4}{m\pi}\cos(m\pi 1) = 0$$

and finally

$$4\frac{\sin(m\pi)}{m^2\pi^2} = 0.$$

$$u^0(x) = \frac{8}{\pi^2}\left(\frac{\sin(\pi x)}{1^2} - \frac{\sin(3\pi x)}{3^2} + \frac{\sin(5\pi x)}{5^2}\dots\right)$$

Here is a rough approximation of $u^0(x)$, Fourier sine series truncated after 3 terms:



The "hat function" is between $x = 0$ and $x = 1$. But it converges very slowly.

$$(ii),\ \int_{2p}^{2p+2}\frac{dx}{x^2} > \frac{2}{(2p+1)^2}$$

$$\int_{2p}^{2p+2}\frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{2p}^{2p+2} = -\frac{1}{2p+2} + \frac{1}{2p}$$

15.1 Assignment 42.

Summary

- Mathematical Modelling
- Three-level schemes for heat equation
- Last revision September 30, 2017

Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2};$$

$$x\varepsilon[0,1], \quad t \geq 0,$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = 2x, \ 0 \leq x \leq \frac{1}{2},$$

$$u(x,0) = 2(1-x), \ \frac{1}{2} \leq x \leq 1.$$

Taylor-series expansions

Three-level scheme, explicit

Three-level scheme, Crank-Nicholson

Suppose

$$0 = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = 1$$

is an equidistant partition of $[0,1]$,

$$\Delta x = x_{i+1} - x_i, \quad i = 0, 1, \ldots n-1.$$

Let Δt be the time increment between time levels. Suppose further that $f(x,t)$ is m times continuously differentiable in the neighbourhood of (x_0, t_0), $m \geq 5$. Then consider the computational molecule

$$\begin{array}{ccccc} (x_0-\Delta x, t_0+\Delta t) & - & (x_0, t_0+\Delta t) & - & (x_0+\Delta x, t_0+\Delta t) \\ & & | & & \\ & & (x_0, t_0) & & \\ & & | & & \\ (x_0-\Delta x, t_0-\Delta t) & - & (x_0, t_0-\Delta t) & - & (x_0+\Delta x, t_0-\Delta t). \end{array}$$

Next, we develop the Taylor-expansions. All derivatives are at (x_0, t_0). Higher order derivatives are denoted by h.o.t.

$$f(x_0+\Delta x, t_0+\Delta t) = f(x_0,t_0)$$

$$+\Delta x f_x + \Delta t f_t$$

$$+\frac{1}{2}\Delta x^2 f_{xx} + \Delta x \Delta t f_{xt} + \frac{1}{2}\Delta t^2 f_{tt}$$

$$+\frac{1}{6}\Delta x^3 f_{xxx} + \frac{1}{2}\Delta x^2 \Delta t f_{xxt} + \frac{1}{2}\Delta x \Delta t^2 f_{xtt} + \frac{1}{6}\Delta t^3 f_{ttt}$$

$$+\frac{1}{24}\Delta x^4 f_{xxxx} + \frac{1}{6}\Delta x^3 \Delta t f_{xxxt} + \frac{1}{4}\Delta x^2 \Delta t^2 f_{xxtt} + \frac{1}{6}\Delta x \Delta t^3 f_{xttt} + \frac{1}{24}\Delta x \Delta t^4 f_{tttt}$$

$$+h.o.t.$$

$$f(x_0, t_0+\Delta t) = f(x_0,t_0)$$

$$+\Delta t f_t + \frac{1}{2}\Delta t^2 f_{tt} + \frac{1}{6}\Delta t^3 f_{ttt} + \frac{1}{24}\Delta x \Delta t^4 f_{tttt} + h.o.t.$$

$$f(x_0-\Delta x, t_0+\Delta t) = f(x_0,t_0)$$

$$-\Delta x f_x + \Delta t f_t$$

$$+\frac{1}{2}\Delta x^2 f_{xx} - \Delta x \Delta t f_{xt} + \frac{1}{2}\Delta t^2 f_{tt}$$

$$-\frac{1}{6}\Delta x^3 f_{xxx} + \frac{1}{2}\Delta x^2 \Delta t f_{xxt} - \frac{1}{2}\Delta x \Delta t^2 f_{xtt} + \frac{1}{6}\Delta t^3 f_{ttt}$$

$$+\frac{1}{24}\Delta x^4 f_{xxxx} - \frac{1}{6}\Delta x^3\Delta t f_{xxxt} + \frac{1}{4}\Delta x^2\Delta t^2 f_{xxtt} - \frac{1}{6}\Delta x\Delta t^3 f_{xttt} + \frac{1}{24}\Delta x\Delta t^4 f_{tttt}$$

$$+h.o.t.$$

$$f(x_0 + \Delta x, t_0 - \Delta t) = f(x_0, t_0)$$

$$+\Delta x f_x - \Delta t f_t$$

$$+\frac{1}{2}\Delta x^2 f_{xx} - \Delta x\Delta t f_{xt} + \frac{1}{2}\Delta t^2 f_{tt}$$

$$+\frac{1}{6}\Delta x^3 f_{xxx} - \frac{1}{2}\Delta x^2\Delta t f_{xxt} + \frac{1}{2}\Delta x\Delta t^2 f_{xtt} - \frac{1}{6}\Delta t^3 f_{ttt}$$

$$+\frac{1}{24}\Delta x^4 f_{xxxx} - \frac{1}{6}\Delta x^3\Delta t f_{xxxt} + \frac{1}{4}\Delta x^2\Delta t^2 f_{xxtt} - \frac{1}{6}\Delta x\Delta t^3 f_{xttt} + \frac{1}{24}\Delta x\Delta t^4 f_{tttt}$$

$$+h.o.t.$$

$$f(x_0, t_0 - \Delta t) = f(x_0, t_0)$$

$$-\Delta t f_t + \frac{1}{2}\Delta t^2 f_{tt} - \frac{1}{6}\Delta t^3 f_{ttt} + \frac{1}{24}\Delta x\Delta t^4 f_{tttt} + h.o.t.$$

$$f(x_0 - \Delta x, t_0 - \Delta t) = f(x_0, t_0)$$

$$-\Delta x f_x - \Delta t f_t$$

$$+\frac{1}{2}\Delta x^2 f_{xx} + \Delta x\Delta t f_{xt} + \frac{1}{2}\Delta t^2 f_{tt}$$

$$-\frac{1}{6}\Delta x^3 f_{xxx} - \frac{1}{2}\Delta x^2\Delta t f_{xxt} - \frac{1}{2}\Delta x\Delta t^2 f_{xtt} - \frac{1}{6}\Delta t^3 f_{ttt}$$

$$+\frac{1}{24}\Delta x^4 f_{xxxx} + \frac{1}{6}\Delta x^3\Delta t f_{xxxt} + \frac{1}{4}\Delta x^2\Delta t^2 f_{xxtt} + \frac{1}{6}\Delta x\Delta t^3 f_{xttt} + \frac{1}{24}\Delta x\Delta t^4 f_{tttt}$$

$$+h.o.t.$$

15.2 Assignment 43.

Summary

- Pólya - Szegő
- Problems on the Integral Parts of Numbers (Part VIII)
- Last revision September 30, 2017

Definitions

Let x be a number. Denote by $[x]$ the *integral part* of x that is the integer that satisfies the inequality

$$x - 1 < [x] \leq x < [x] + 1$$

Examples:

$$[\pi] = 3,\ [2] = 2,\ [-0.53] = -1$$

1.

Let n be an integer and x arbitrary. We then have

$$[x + n] = [x] + n.$$

Proof:

$$x + n - 1 < [x + n] \leq x + n < [x + n] + 1.$$

$$x - 1 < [x + n] - n \leq x.$$

By definition,

$$x - 1 < [x] \leq x.$$

Between $x-1$ and x there is one and only one integer, $[x]$. But $[x+n]-n$ is also an integer between $x - 1$ and x. Therefore the two should be the same

$$[x + n] - n = [x]$$

hence

$$[x + n] = [x] + n. \surd$$

2.

In the expansion of the determinant of $n-th$ order, the product of the elements in the secondary diagonal has the sign $(-1)^{[\frac{n}{2}]}$.

Proof:

$$\det A = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} = \sum (-1)^{t(j)} a_{1,j_1} a_{2,j_2} \dots a_{n,j_n}$$

where $t(j)$ is the number of inversions in the permutation $t(j) = (j_1, j_2, \dots, j_n)$. The secondary diagonal is

$$a_{1,n} a_{2,n-1} \dots a_{n,1}$$

and

$$t(j) = (n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$$

by the well-known summation formula. Thus we are required to show that

$$\frac{n(n-1)}{2} \equiv \left[\frac{n}{2}\right], \text{ (modulo 2).}$$

Next, we consider four cases, $n = 0, 1, 2, 3$.

$$n = 0; \ \frac{0(0-1)}{2} \equiv \left[\frac{0}{2}\right] \equiv 0 \text{ (modulo 2).}$$

$$n = 1; \ \frac{1(1-1)}{2} \equiv \left[\frac{1}{2}\right] \equiv 0 \text{ (modulo 2).}$$

$$n = 2; \ \frac{2(2-1)}{2} \equiv \left[\frac{2}{2}\right] \equiv 1 \text{ (modulo 2).}$$

$$n = 3; \ \frac{3(3-1)}{2} \equiv \left[\frac{3}{2}\right] \equiv 1 \text{ (modulo 2).}$$

The required congruence holds for $n = 0, 1, 2, 3$.

Take $n+4$.

$$\frac{(n+4)(n+3)}{2} - \frac{n(n-1)}{2} = \frac{8n+12}{2} = 4n+6 \equiv 0 \text{ (modulo 2)},$$

the difference is an even number. Therefore

$$\frac{(n+4)(n+3)}{2} \equiv \frac{n(n-1)}{2} \equiv \left[\frac{n}{2}\right] \text{ (modulo 2)}$$

and the sign of the product is $(-1)^{[\frac{n}{2}]}$ for every n. $\surd$

3.

We have

$$[2x] - 2[x] = 0 \ \textit{or} \ 1$$

according as

$$x - [x] < \frac{1}{2} \ \textit{or} \ \geq \frac{1}{2}.$$

Proof:

Recall that by definition

$$x - 1 < [x] \leq x < [x] + 1$$

$[x]$ is the greatest integer not exceeding x, which can also be termed *left-adjacent* to x.

Case i)

$$0 \leq x - [x] < \frac{1}{2}$$

$$0 \leq 2x - 2[x] < 1$$

The greatest integer not exceeding $2x - 2[x]$ cannot be 1 because 1 exceeds $2x - 2[x]$. (Check equation above.) So the required integer integer is less than 1; 0 is less than 1, furthermore 0 less than or equal to $2x - 2[x]$. Can the greatest integer not exceeding $2x - 2[x]$ be other than 0 ? No, because there is no integer between 0 and 1. Therefore

$$[2x - 2[x]] = 0$$

and by 1.

$$[2x - 2[x]] = [2x] - 2[x] = 0.$$

Case ii)

$$x - [x] \geq \frac{1}{2}$$

$$2 > 2x - 2[x] \geq 1$$

$$[2 > 2x - 2[x]] = 1$$

By 1. again

$$[2 > 2x - 2[x]] = [2x] - 2[x] = 1. \surd$$

4.

If $0 < \alpha < 1$, *then we have*

$$[x] - [x - \alpha] = 0 \ \textit{or} \ 1$$

according as

$$x - [x] \geq \alpha \ \textit{or} \ < \alpha.$$

Proof:

Write

$$x - [x] = \{x\}$$

where $\{x\}$ is the *fractional part* of x. Clearly

$$x = [x] + \{x\}$$

and $\{x\}$ is non-negative.

$$[x] - [x - \alpha] = [x] - [[x] + \{x\} - \alpha].$$

Apply 1 to $[[x] + (\{x\} - \alpha)]$, $[x]$ is integer, $(\{x\} - \alpha)$ is arbitrary. If

$$0 \leq \{x\} - \alpha < \{x\} < 1$$

then

$$[x] - [x - \alpha] = 0.$$

If

$$-1 < \{x\} - \alpha < 0$$

then

$$[[x] + \{x\} - \alpha] = [x] - 1,$$

and

$$[x] - [x - \alpha] = 1. \surd$$

5.

Let x be a number that does not lie at the midpoint between two consecutive integers. Express the integer nearest to x in terms of the symbol [ ].

Write

$$\beta = x - [x];\ 0 \leq \beta < 1,$$

where β is the fractional part of x. Observe that

$$0 \leq 2\beta < 2.$$

Therefore $[2\beta] = 1$ if $\beta > \dfrac{1}{2}$, and $[2\beta] = 0$ if $\beta < \dfrac{1}{2}$.

In the first case, $[x] + 1$ is the closest integer to x,

$$[x] + 1 = [x] + [2\beta] = [x] + [2(x - [x])] = [x] + [2x - 2[x]] = [x] + [2x] - 2[x].$$

In the second case $[x]$ is the closest integer to x,

$$[x] = [x] + [2\beta] = [x] + 2(x - [x]) = 2x - [x].$$

Therefore the integer nearest to x in terms of of the symbol [ ] is $2x - [x]\surd$

15.3 Assignment 44.

Summary

- Mathematical Modelling
- *Blasius equation: Westbrook*
- Last revision September 30, 2017

Problem 1.

Apply the method of stretching to the equation

$$\Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} - \Psi_{yyy} = 0 \qquad (15.9)$$

in order to reduce it to an ordinary differential equation.

Equation 15.9 occurs in boundary layer theory for flow along a flat plate. It is equivalent to the following set of equations:

$$uu_x + u_y = u_{yy} \qquad (15.10)$$

$$u_x + v_y = 0 \qquad (15.11)$$

where $u = \Psi_y$, $v = -\Psi_x$; and Ψ is the stream function. Let

$$\bar{x} = a^n x,\ \bar{y} = a^m x,\ \bar{u} = a^p x,\ \bar{v} = a^q x$$

This is a one parameter stretching transform. Our objective is to determine

$$n, m, p, q;$$

a similarity variable η, and an ordinary diff. eq. - simultaneously.

1st. Claim

If

$$-q = m; \quad n - 2m = p$$

then

$$\bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} = \bar{u}_{\bar{y}\bar{y}}$$

$$\bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} = 0$$

satisfy the constant conformally invariant condition.

Proof of 1st. Claim

$$\bar{u}_{\bar{x}} = (a^p u)_x \frac{\partial x}{\partial \bar{x}} = a^p u_x a^{-u} = a^{-u+p}$$