Pinter Consulting
Quarterly Reports
New Series No. 9.
Rough Copy

J K Pinter, Dr.Tech.

August 1, 2015

Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

Introduction

Pinter Consulting of Calgary, Alberta practices Mathematics, promotes clear thinking and offers Consultations, Tutorials and Seminars in Mathematics.

Contents

Chapter 9

Proceedings

9.1 Summary of Current Report

- **Private study for professional development:**
- Records of activities at Pinter Consulting
- Collection of problems with our own solutions .
- GRE Mathematics Test (Resource: www.ets.org/gre)
- In compliance with Terms of Use "fair use for educational purposes"
- Continuous improvement, corrections and last revisionAugust 1, 2015.

9.2 Solutions 34-44.

- Mathematics Test (GRE)
- *Spartan Old School Tutorials*
- Last revision August 1, 2015

Problems

34. Find the minimal distance between any point on the sphere

$$S_1 = \{(x, y, z) : (x-2)^2 + (y-1)^2 + (z-3)^2 = 1\}$$

and any point on the sphere

$$S_2 = \{(x, y, z) : (x+3)^2 + (y-2)^2 + (z-4)^2 = 4\}.$$

Solution: In the *3-dimensional Euclidean space* the distance between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_3)$ is given by

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The distance between any two sets S_1 and S_2 is defined by

$$d(S_1, S_2) = \inf\{d(P, Q) : P \epsilon S_1; Q \epsilon S_2\}.$$

Spheres are compact sets therefore

$$d(S_1, S_2) = \min\{d(P, Q) : P \epsilon S_1; Q \epsilon S_2\}.$$

We can find the minimum by *geometric* means. The two centers are

$$O_1 = (2, 1, 3), \quad O_2 = (-3, 2, 4),$$

respectively. The distance between them is

$$d(O_1, O_2) = \sqrt{(-3-2)^2 + (2-1)^2 + (4-3)^2} = \sqrt{27}.$$

Less the two *radii*

$$\sqrt{27} - 1 - 2 = 3(\sqrt{3} - 1).$$

35. At a banquet, 9 women and 6 men are to be seated in a row of 15 chairs. If the entire seating arrangement is to be chosen at random, what is the probability that all men will be seated next to each other in 6 consecutive position?

Solution: Given two sets

$$\{F_1, F_2, \ldots, F_9\}; \quad \{M_1, M_2, \ldots, M_6\}$$

we have to count *favourable arrangements* versus *all arrangements* . The latter is equal to 15! the permutation of 15 elements, (individuals). The former is equal to 10!, the permutation of 10 elements

$$\{F_1, F_2, \ldots, F_9, M\}$$

where M denotes the six men lumped together, multiplied by 6!, distinguishing each permutation of the elements (individuals) within the set M. Therefore the sought probability is

$$\frac{6!10!}{15!}.$$

36. Let M be a 5×5 real matrix. Exactly four of the following five conditions are equivalent.

(A) For two distinct column vectors u, v of M, the set u, v is linearly independent.

(B) The homogeneous system $Mu = 0$ has only the trivial solution.

(C) The system of equations $Mx = b$ has a unique solution for b .

(D) The determinant of M is nonzero.

(E) There exists a 5×5 real matrix N such that NM is the 5×5 identity matrix.

Solution: Clearly, conditions B,C,D,E are equivalent and express the fact that matrix M is non-singular, or has full rank (5), real matrix $N = M^{-1}$. The matrix in (A) could be deficient

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

37. In the complex z-plane, the set of points satisfying equation $z^2 = |z|^2$ is a line, in fact, it is the real line.

Proof: Write

$$z = r \exp{(\imath\phi)}; \quad r > 0; \quad 0 \le \phi < 2\pi.$$

$$z^2 = r \exp{(\imath\phi)} * r \exp{(\imath\phi)} = r^2 \exp{(\imath 2\phi)}.$$

$$|z|^2 = r^2$$

$$\exp{(\imath 2\phi)} = 1 \Rightarrow \phi = 0; \phi = \pi$$

38. Let A and B be non-empty subsets of R and let $f : A \to B$ be a function. If $C \subseteq A$ and $D \subseteq B$, then
(A) $C \subseteq f^{-1}(f(C))$ is TRUE,
(B) $D \subseteq f(f^{-1}(D))$ is FALSE,
(C) $f^{-1}(f(C)) \subseteq C$ is FALSE,
(D) $f^{-1}(f(C)) = f(f^{-1}(D))$ is FALSE,
(E) $f(f^{-1}(D)) = f^{-1}(D)$ is FALSE.

Proof: Let f be any function from set A to set B. Note that f can be neither *onto* nor *one-to-one*. For a subset S of A write $f(S)$ for the set of all $f(s)$ with $s \epsilon S$. For a subset of T of B, we define $f^{-1}(T)$ to be the set of all a in A with $f(a) \epsilon T$, and we call $f^{-1}(T)$ the *inverse image* of T; some other texts call it *pre-image*. Thus we have extended the definition of f on elements to power sets

$$f : P(A) \to P(B).$$

What is the inverse image of a typical element b in B? It is a subset of A, the $\emptyset$ if b is not in the image (range) of f.

Looking at claim(A), let c be a typical element in C, $C \neq \emptyset$

$$f : c \mapsto f(c) \ \epsilon \ f(C)$$

$$f^{-1}(f(C)) \supseteq f^{-1}(f(c)) \supseteq \{c\}$$

$$f^{-1}(f(C)) \supseteq C.$$

Therefore claim (A) is TRUE.

As for claim(B), suppose $D \neq \emptyset$ is not in the range of f. Then $f^{-1}(D) = \emptyset$ and $f(\emptyset)$ assigns to every element of $\emptyset$ an element in B. But $\emptyset$ has no elements so there is no assignments in $f(\emptyset)$, $f(\emptyset) = \emptyset$, $D \subseteq \emptyset$, a contradiction.

Suppose, if possible, that claim(C) is true:

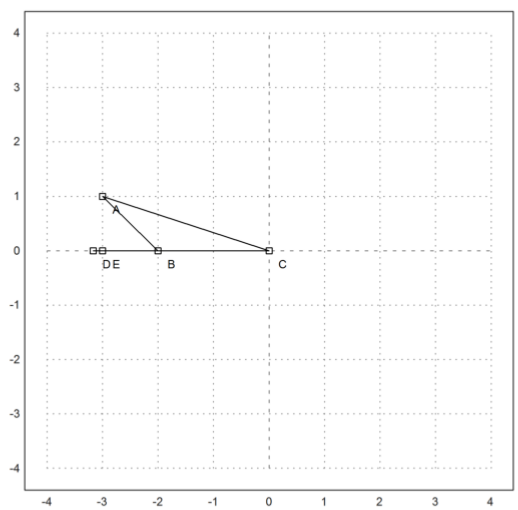
$$C \supseteq f^{-1}\left(f(C)\right).$$

Then by claim(A)

$$C = f^{-1}\left(f(C)\right),$$

or f is *one-to-one* . Contradiction, f is not assumed to be such. Therefore claim(C) is FALSE.

We can dismiss claims(D) and (C) as ABSURD (hence FALSE), because the right and left hand sides are in different sets.



39.

Let ABC be an obtuse triangle, angle $\beta = 110\,\text{deg}$ at vertex B. Let $AB = 1$, $BC = r$, $CA = s$. Then

$$\lim(r - s) = \text{ a positive number less than 1}$$

as $s \to \infty$ and $r \to \infty$.

Proof: Extend BC past B and call the point D for which $DC = AC$. Then DA is an arc of the circle centered at C with radius s (not shown). Drop a perpendicular from A to DB and call the point E. Clearly, $EB = \cos(70\,\text{deg}), 0 < EB < 1$.

$$\lim(r - s) = \lim(AC - BC) = \lim DB > EB$$

As $s \to \infty$ and $r \to \infty$ $DB \to EB$ but DB is bounded from below by EB and so the geometric limit exists.

40. In the rings of continuous real-valued functions on $[0, 1]$ it is possible for the product of two non-zero elements be zero.

Proof: Let $f(x)$ and $g(x)$ be two non-zero elements of the ring in question. The *support* of f, Supp(f), is the set of x for which $f(x) \neq 0$. Set the supports of f and g such

$$\text{Supp(f)} \cap \text{Supp(g)} = \emptyset$$

then the pointwise product is zero, $f(x) * g(x) = 0$.

41. Let C be the circle $x^2 + y^2 = 1$ oriented counterclockwise in the $xy-$plane.

$$\oint_C (2x - y)dx + (x + 3y)dy = 2\pi.$$

Proof: Substitution leads to parametric integral.

$$x = \cos(t), \quad dx = -\sin(t)dt$$

$$y = \sin(t), \quad dy = \cos(t)dt$$

$$\int_0^{2\pi} (2\cos(t) - \sin(t))(-\sin(t)dt) + \int_0^{2\pi} (\cos(t) + 3\sin(t))\cos(t)dt =$$

$$-\int_0^{2\pi} 2\cos(t)\sin(t)dt + \int_0^{2\pi} \sin^2(t)dt + \int_0^{2\pi} \cos^2(t)dt + \int_0^{2\pi} 3\sin(t)\cos(t)dt =$$

$$+\int_0^{2\pi} \sin(t)\cos(t)dt + \int_0^{2\pi} (\sin^2(t) + \cos^2(t))dt.$$

$$\int_0^{2\pi} \sin(t)\cos(t)dt = \left[\frac{\sin^2(t)}{2}\right]_0^{2\pi} = 0$$

$$\int_0^{2\pi} (\sin^2(t) + \cos^2(t))dt = \int_0^{2\pi} dt = 2\pi.$$

42. Suppose X is a discrete random variable on the set of positive integers such that for each positive integer n, the probability that $X = n$ is $\frac{1}{2^n}$. If Y is a random variable with the same distribution and X and Y are independent, the probability that the value of at least one of the variables X and Y is greater than 3 is $\frac{15}{64}$.

Proof:

$$Pr(X = 1) = \frac{1}{2}; \quad Pr(X = 2) = \frac{1}{4}; \quad Pr(X = 3) = \frac{1}{8};$$

$$Pr(Y = 1) = \frac{1}{2}; \quad Pr(Y = 2) = \frac{1}{4}; \quad Pr(Y = 3) = \frac{1}{8};$$

$$Pr(X \leq 3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

$$Pr(Y \leq 3) = Pr(X \leq 3)$$

$$Pr(X \leq 3 \wedge Y \leq 3) = Pr(X \leq 3) * Pr(Y \leq 3) = \frac{49}{64}$$

$$Pr(X > 3 \vee Y > 3) = 1 - Pr(X \leq 3) * Pr(Y \leq 3) = 1 - \frac{49}{64} = \frac{15}{64}.$$

43. If $z = \exp(\frac{2\pi\imath}{5})$, then

$$1 + z + z^2 + z^3 + 5z^4 + 4z^5 + 4z^6 + 4z^7 + 4z^8 + 5z^9 = -5\exp(\frac{3\pi\imath}{5}).$$

Proof: Write

$$Z = 1 + z + z^2 + z^3 + 5z^4 + 4z^5 + 4z^6 + 4z^7 + 4z^8 + 5z^9.$$

$$Z = (1 + z + z^2 + z^3 + z^4) + (4z^4 + 4z^5 + 4z^6 + 4z^7 + 4z^8) + 5z^9 =$$

$$(1 + z + z^2 + z^3 + z^4) + 4z^4(1 + z + z^2 + z^3 + z^4) + 5z^9.$$

Roots of unity, geometric series

$$1 + z + z^2 + z^3 + z^4 = \frac{z^5 - 1}{z - 1} = \frac{\exp(2\pi\imath) - 1}{\exp(\frac{2\pi\imath}{5}) - 1} = 0$$

$$Z = 5z^9$$

$$Z = 5\exp(9\frac{2\pi\imath}{5}) = 5\exp(\frac{18\pi\imath}{5}) = 5\exp(3\pi\imath + \frac{3\pi\imath}{5}) = 5\exp(3\pi\imath) * \exp(\frac{3\pi\imath}{5})$$

$$Z = -5\exp(\frac{3\pi\imath}{5}).$$

44. A fair coin is to be tossed 100 times, with each toss resulting in a head or a tail. If H is the total number of heads and T is the total number of tails, approximate (or estimate) the probability of the event $H \geq 48 \wedge T \geq 48$.

Solution: Simple alternative, two outcomes, head (A) or tail ($\overline{A}$) with probabilities $p = 0.5$ and $q = 1 - p = 0.5$, independent trials. In general, let ξ be a discrete random variable with binomial distribution. If $\xi = k,\ k = 1, 2 \ldots n$ denotes the event that 'head' (A) occured exactly k times out of n trials then the probability is

$$Pr(\xi = k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k}, \quad (k = 0, 1, \ldots n)$$

the expected value (the mean) μ is

$$\mu = \sum_{k=0}^{n} k \left(\begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k} = np$$

and the square of standard deviation σ is

$$\sigma^2 = \sum_{k=0}^{n} (k-np)^2 \binom{n}{k} = npq.$$

The probabilities of binomial distributions for sufficiently large n can be approximated by the *normal distribution.* This fact has experimantal as well as theoretical verification, see *Galton's board; de Moivre, Laplace, Gauss.*Set

$$np + a\sqrt{npq} + \frac{1}{2} = A$$

$$np + b\sqrt{npq} - \frac{1}{2} = B$$

A, B integers, $A < B$, $p = q$ then

$$\sum_{A \le k \le B} \binom{n}{k} p^k q^{n-k} = \frac{1}{\sqrt{2\pi}} \int_a^b \exp\left(-\frac{x^2}{2}\right) dx + O\left(\frac{1}{n}\right)$$

Let us calculate the probability

$$\xi = k,\ 48 \le k \le 52,\ n = 100,\ p = q = 0.5$$

$$48 = 50 + a\sqrt{25} + 0.5 = 50.5 + 5a \to a = -0.5$$

$$52 = 50 + b\sqrt{25} - 0.5 = 49.5 + 5b \to b = 0.5$$

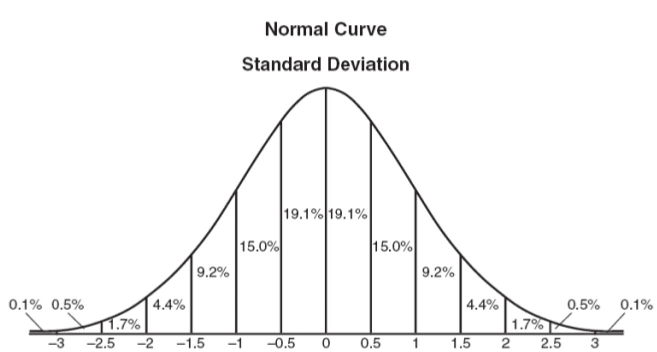
$$\frac{1}{\sqrt{2\pi}} \int_{-0.5}^{0.5} \exp\left(-\frac{x^2}{2}\right) dx = 0.383.$$

Another approximation (which uses Sterling's formula) yields

$$\sum_{48 \le k \le 52} \binom{n}{k} p^k q^{n-k} = 0.3827.$$

Directly from Gaussian normal curve we get

$$0.191 + 0.191 = 0.382$$



9.3 Solutions 45-55.

- Mathematics Test (GRE)
- *Spartan Old School Tutorials*
- Last revision August 1, 2015

Problems

45. A circular region is divided by 5 radii into 5 congruent sectors. Call the sectors A,B,C,D,E; in clockwise direction. 21 points are chosen in the circular region, none of which is on any of the 5 radii. Check the following statements:

(I) Some sector contains at least 5 points.

(II) Some sector contains at most 3 points.

(III) Some pair of adjacent sectors contains a total of at least 9 of the points.

Proof: (I) TRUE. Suppose - if possible - that none of the sectors have more than 4 points. Let each sector have exactly 4 points. Then there are only 20 points chosen. This is a contradiction, therefore there is at least one sector with 5 points.

(II) FALSE. This statement means that in every eligible selection of 21 points there is at least one sector with 3 or less points. Just consider the selection in (I): there are sectors with more than 3 points, (actually all of them).

(III) TRUE. Suppose - if possible - that none of the pairs of adjacent sectors have more than 8 points. List the adjacent pairs of sectors: AB; BC; CD; DE; EA. If all pairs have 8 or less points then we have at most $5 \times 8 = 40$ points, each point counted exactly twice. But this is a countradiction

$$\frac{40}{2} < 21.$$

Therefore at least one pair of adjacent sectors have 9 or more points.

46. Let G be the group of complex numbers $\{1, \imath, -1, -\imath\}$, *4-th roots of unity* under multiplication. The following statements are true about the homomorphisms of G into itself.

(I) $z \mapsto \bar{z}$ *complex conjugation* is one such homomorphism.

(II) $z \mapsto z^2$ is another such homomorphism.

(III) For every such homomorphism, there is an integer k such that the homomorphism has the form $z \mapsto z^k$.

Discussion: We start with (III): Multiplication Table for group G of complex numbers $\{1, \imath, -1, -\imath\}$, *4-th roots of unity* :

	1	$\imath$	-1	$-\imath$
1	1	$\imath$	-1	$-\imath$
$\imath$	$\imath$	-1	$-\imath$	1
- 1	-1	-$\imath$	1	$\imath$
-$\imath$	-$\imath$	1	$\imath$	-1

Multiplicative unit is 1. Group G is cyclic, generated by $\imath$

$$\imath = \imath;\ \imath^2 = -1;\ \imath^3 = -\imath;\ \imath^4 = 1.$$

The function $f : G \longrightarrow G$ is a *homomorphism* if

$$f(ab) = f(a)f(b);\ \forall a, b\ \epsilon G.$$

The homomorphic image of a cyclic group is cyclic.

$$f(\imath);\ f(\imath^2) = f^2(\imath);\ f(\imath^3) = f^3(\imath);\ f(\imath^4) = f^4(\imath).$$

Thus $Im(f)$ is generated by $f(\imath)$. But $f : G \longrightarrow G$ therefore $f(\imath) = \imath^k$ for some k, $k = 1, 2, 3, 4$, moreover every other element of $Im(f)$ is of the form of $\imath^{m*k}$ where m is a positive integer.

(I): Consider *complex conjugation :*

$$f(\imath) = \imath^3 = -\imath,\ f(\imath^2) = \imath^6 = -1,\ f(\imath^3) = \imath^9 = \imath,\ f(\imath^4) = \imath^{12} = 1.$$

This is an *isomorphism* . (Note $f(\imath) = \imath^{-1}$).

(II): Next, check $f(i) = i^2$. This, too, is a homomorphism, in fact, an *endomorphism,* $Im(f) = \{1, -1\}$.

47. Let $\bar{F}$ be a constant unit force that is parallel to the vector $(-1, 0, 1)$ in the xyz-plane. What is the work done by $\bar{F}$ on a particle that moves along the path given by (t, t^2, t^3) between time $t = 0$ and $t = 1$?

Solution: Constant unit force

$$\bar{F} = (\frac{-1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}).$$

Tangent to parametric curve (t, t^2, t^3)

$$(1,\ 2t,\ 3t^2).$$

Scalar product of constant unit force and tangent integrated over $[0, 1]$

$$\frac{1}{\sqrt{2}} \int_0^1 (-1 + 3t^2)dt = \frac{1}{\sqrt{2}} \left[-t + t^3\right]_0^1 = 0.$$

Work done equals zero.

48. Theorem: If f and f' are both strictly increasing real-valued functions on the interval $(0, \infty)$, then

$$\lim_{x \to \infty} = \infty$$

Proof: By the Mean Value Theorem (a.k.a. Lagrange Theorem), there is a c_1 in the interval $(1, 2)$ such that

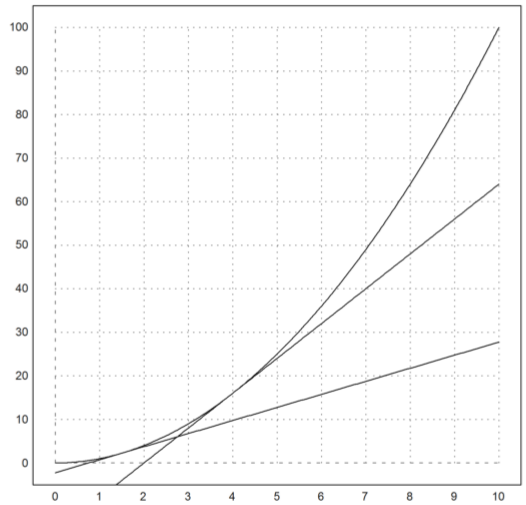
$$f'(c_1) = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1) > 0.$$

For each $x > 2$, there is a c_x in $(2, x)$ again, by the Mean Value Theorem, such that

$$\frac{f(x) - f(2)}{x - 2} = f'(c_x) > f'(c_1),$$

$$f(x) = f(2) + (x - 2)f'(c_x) > f(2) + (x - 2)f'(c_1),$$

which proves the theorem. An illustration of the Theorem follows:



49. Up to isomorphism, how many additive Abelian groups G of order 16 have the property that

$$x + x + x + x = 0$$

for each x in G?

Solution: Three groups. They are

$$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$$

$$Z_2 \oplus Z_2 \oplus Z_4$$

$$Z_4 \oplus Z_4.$$

Group $Z_2 \oplus Z_2$ is the *Klein* 4 *Group*, *symmetries of the rectangle,* which is not isomorphic to Z_4.

Discussion: First, we state that it is sufficient to consider Z_n, $n = 16$ the Abelian (commutative) group of integers modulo 16. Z_{16} is cyclic, it is generated by remainder 1, all of its subgroups are cyclic.

Fundamental (omnibus) theorem of finite Abelian groups: Every finite Abelian group G which is not a zero group can be decomposed into a direct sum of primary cyclic subgroups *(Kurosh).*

Primary means prime power. So in case $n = 16$, excluding the trivial subgroups, we have Z_2, Z_4, Z_8. However, Z_8 does not meet the condition that each $4x$ vanish. Thus we are left with Z_2, Z_4.

50. Let A be a 2×2 matrix. Then

(A) All of the entries of A^2 are nonnegative. FALSE.

$$A = \begin{bmatrix} 1 & 1 \\ -117 & 0 \end{bmatrix} \; A^2 = \begin{bmatrix} 1 & 1 \\ -117 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -117 & 0 \end{bmatrix} = \begin{bmatrix} -116 & X \\ X & X \end{bmatrix}$$

(B) The determinant of A^2 is nonnegative. TRUE.

$$\det(A^2) = \det(A) * \det(A) \geq 0.$$

(C) If A has two distinct eigenvalues, A^2 has two distinct eigenvalues. FALSE.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \; \lambda_1 = 1, \; \lambda_2 = -1,$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; \lambda_1 = \lambda_2 = 1.$$

51. If $ent(x)$ denotes the greatest integer not exceeding x, (ent=entier) then

$$\int_0^\infty ent(x) \exp(-x) dx = \frac{1}{e-1}$$

Proof:

$$\int \exp(-x) dx = -\exp(-x) + C.$$

Moreover, the *entier* function has jumps at $x = n$;

$$\int_{n-1}^{n} (n-1) \exp(-x) dx = (n-1)(-\exp(-n)) - (n-1)(-\exp(-n+1)).$$

$$\int_{n}^{n+1} n \exp(-x) dx = n(-\exp(-n-1)) - n(-\exp(-n)).$$

Notice that there are $(-\exp(-n))$ terms in both definite integrals. This suggests a *telescoping sum* so let us write out some of the first jumps,

$$ent(x) = \begin{cases} 0; & 0 \leq x < 1 \\ 1; & 1 \leq x < 2 \\ 2; & 2 \leq x < 3 \\ 3; & 3 \leq x < 4 \\ 4; & 4 \leq x < 5 \end{cases}$$

$$\begin{aligned} \int_0^5 ent(x)\exp(-x)dx &= \int_0^1 0 * \exp(-x)dx \\ &+ \int_1^2 1 * \exp(-x)dx \\ &+ \int_2^3 2 * \exp(-x)dx \\ &+ \int_3^4 3 * \exp(-x)dx \\ &+ \int_4^5 4 * \exp(-x)dx \end{aligned}$$

$$\int_0^1 0 * \exp(-x)dx = 0$$

$$\int_1^2 1 * \exp(-x)dx = (-\exp(-2)) - (-\exp(-1))$$

$$\int_2^3 2 * \exp(-x)dx = (-2\exp(-3)) - (-2\exp(-2))$$

$$\int_3^4 3 * \exp(-x)dx = (-3\exp(-4)) - (-3\exp(-3))$$

$$\int_4^5 4 * \exp(-x)dx = (-4\exp(-5)) - (-4\exp(-4))$$

Simplifying

$$\int_0^1 0 * \exp(-x)dx = 0$$

$$\int_1^2 1 * \exp(-x)dx = \exp(-1) - \exp(-2)$$

$$\int_2^3 2 * \exp(-x)dx = 2\exp(-2) - 2\exp(-3)$$

$$\int_3^4 3 * \exp(-x)dx = 3\exp(-3) - 3\exp(-4)$$

$$\int_4^5 4 * \exp(-x)dx = 4\exp(-4) - 4\exp(-5)$$

$$\int_0^1 + \int_1^2 + \ldots + \int_4^5 = \exp(-1) + \exp(-2) + \exp(-3) + \exp(-4) - 4\exp(-5)$$

We can collect our results :

$$\int_0^5 ent(x)\exp(-x)dx = \text{finite geometric series} + \text{remainder}.$$

What remains to show is that the finite geometric series turns into convergent infinite geometric series, summable; and the remainder goes to zero as $x \to \infty$.

$$\sum_{k=1}^{\infty} \exp(-k) = \exp(-1)\sum_{k=0}^{\infty} \exp(-k) = \frac{\exp(-1)}{1 - exp(-1)} = \frac{1}{exp(1) - 1}.$$

52. Check the following statements, if A is a subset of the real line R and A contains each rational number.

(A) If A is open, then $A = R$.
(B) If A is closed, then $A = R$.
(C) If A is uncountable, then $A = R$.
(D) If A is uncountable, then A is open.
(E) If A is countable, than A is closed.

Solution: (A) FALSE. Let $A = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$, $A \neq R$.

(B) TRUE. If A is closed, then every Cauchy-convergent series is convergent and has its limit point in A. Let u be irrational. Since A contains each rational number there exists a series of rationals that converges to u. If A is closed then $u \epsilon A$. Therefore A contains all rationals and irrationals.

(C) FALSE. Suppose A has each rationals and $(0, 1)$.

(D) FALSE. Same argument as in (C).

(E) FALSE. The set of rationals is countable but not closed.

53. What is the minimum of the expression $x+4z$ as a function defined on R^3, subject to the constraint $x^2+y^2+z^2 \leq 2$?

Solution: Application of *Lagrange multiplier method.*

$$F(x,z) = x + 4z$$

$$\phi(x,y,z) = x^2+y^2+z^2-2$$

$$G(x,y,z) = F(x,y,z) + \lambda\phi(x,y,z) = x+4z+\lambda(x^2+y^2+z^2-2)$$

$$G_x = 1 + 2\lambda x = 0$$

$$G_y = 2\lambda y = 0$$

$$G_z = 4 + 2\lambda z = 0$$

$$G_\lambda = x^2+y^2+z^2-2 = 0$$

Next, we solve the system of equations

$$G_x = G_y = G_z = G_\lambda = 0.$$

If $\lambda = 0$ then $G_x = G_z = 0$ lead to contradiction. Therefore $\lambda = 0$ is excluded. If $\lambda \neq 0$:

$$4G_x - G_z = 2\lambda(4x - z) = 0$$

$$4x = z; \;\; y = 0.$$

Substitution into the constraint $G_\lambda = 0$ yields

$$x^2 + (4x)^2 = 2$$

$$17x^2 = 2.$$

Next, we take

$$x = -\frac{\sqrt{2}}{\sqrt{17}}$$

with optimism that we shall find a minimum.

$$F(x,y,z) = x + 4z = -\frac{\sqrt{2}}{\sqrt{17}} - 16\frac{\sqrt{2}}{\sqrt{17}} = -17\frac{\sqrt{2}}{\sqrt{17}} = -\sqrt{17}\sqrt{2} = -\sqrt{34}.$$

To complete this minimum problem one has to show that

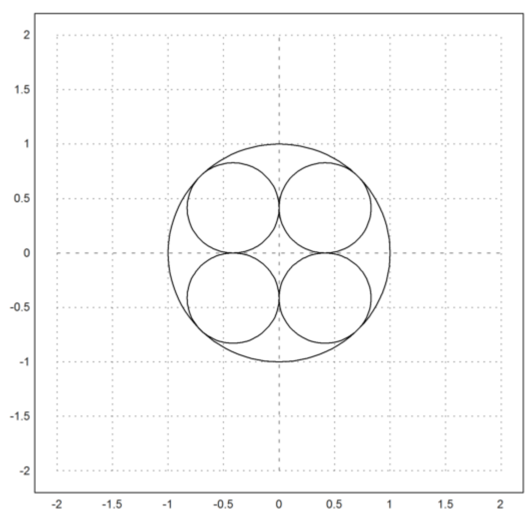
$$P = (-\frac{\sqrt{2}}{\sqrt{17}}, 0, -4\frac{\sqrt{2}}{\sqrt{17}})$$

is not only a *stationary point* but

$$F(P) = -\sqrt{34}$$

is a minimum by investgating the second derivatives; $F(x,y,z)$ takes on its minimum on the boundary.

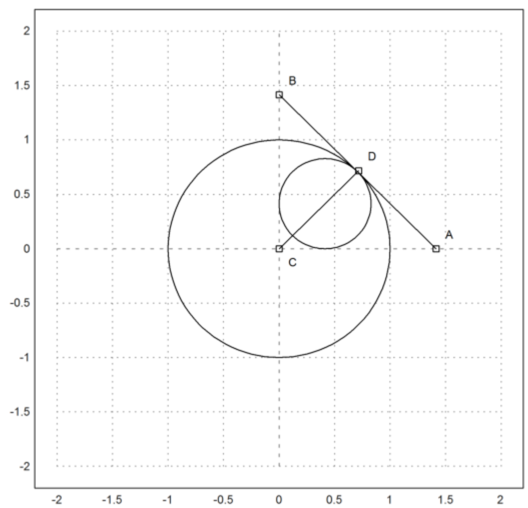
54. Without loss of generality and using obvious symmetries this problem can be simplified. Given the unit circle centered at the origin of the standard Cartesian coordinate system on the Euclidean plane, also given four, smaller, ($r < 1$) congruent circles within the unit circle.(These circles are shaded in the original Figure 1.) Each of the small circles is tangent to the unit circle and two of the other small circles. The coordinate axes are tangent to the small circles:



What is the ratio of the combined area of the small circles to the area of the unit circle?

The problem can be further simplified. It is sufficient to consider the area of one small circle, - say the one in the positive quadrant of the coordinate system - to the area of the intersection of the unit circle and the positive quadrant, $\frac{\pi}{4}$.

Let C denote the center of the unit circle $(0,0)$; and let D be the point of tangential contact of the selected small circle and the unit circle. Segment CD bisects the right angle of the coordinate axes in the positive quadrant.



Draw a tangent to the circles at D, it cuts the $x-axis$ at point A and the $y-axis$ at point B. $\triangle\, CAD$ is a an *isosceles* triangle with right angle at D.

$$CD = DA = 1, \quad CA = \sqrt{2}.$$

Similarly, $\triangle\, CDB$ is a an *isosceles* triangle with right angle at D.

$$CD = DB = 1, \quad CB = \sqrt{2}.$$

Moreover, $\triangle\, CAB$ is a an *isosceles* triangle, too, with right angle at C.

$$CA = CB = \sqrt{2}, \quad AB = 2.$$

The area of $\triangle\, CAB$ is 1. The perimeter is $(2+2\sqrt{2})$. The selected small circle is *inscribed* to $\triangle\, CAB$. The radius r of the inscribed circle is

$$r = \frac{2 \times \text{Area}}{\text{Perimeter}} = \frac{2}{2 \times (1+\sqrt{2})} = \frac{1}{1+\sqrt{2}}.$$

Therefore the area of the selected small circle (hence every congruent small circle) is

$$r^2\pi = \left(\frac{1}{1+\sqrt{2}}\right)^2 \times \pi = \frac{\pi}{(1+\sqrt{2})^2}.$$

The ratio of the areas is

$$\frac{\frac{\pi}{(1+\sqrt{2})^2}}{\frac{\pi}{4}} = \frac{4}{(1+\sqrt{2})^2} = \left(\frac{2}{1+\sqrt{2}}\right)^2.$$

Remark: The theorem about the radius of the inscribed circle can be verified by plotting the centre Q of the small circle to the second figure of our solution and inspecting the areas of $\triangle CQA$, $\triangle AQB$, $\triangle BQC$. The area of $\triangle ABC$ equals the total areas of $\triangle$'s CQA, AQB, BQC.

55. For how many positive integers k does the ordinary decimal representation of the integer $k!$ end in exactly 99 zeros?

Solution: Let p be a prime number. Then the exact power of p that divides $n!$ is given by

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots$$

There are only finitely many non-zero terms in the series. Further, write

$$f(n,p) = \sum_{k=1} \left[\frac{n}{p^k}\right].$$

If p divides n then

$$f(n,p) = f(n+1,p) = \dots = f(n+p-1,p).$$

For illustration, consider $p = 5$, $p = 2$ and $n = 1, 2, \dots 16$

$$f(1,5) = f(2,5) = f(3,5) = f(4,5) = 0$$

$$f(5,5) = f(6,5) = f(7,5) = f(8,5) = f(9,5) = 1$$

$$f(10,5) = \left[\frac{10}{5}\right] = 2$$

$$f(11,5) = \left[\frac{11}{5}\right] = 2$$

$$f(12,5) = \left[\frac{12}{5}\right] = 2$$

$$f(13,5) = \left[\frac{13}{5}\right] = 2$$

$$f(14,5) = \left[\frac{14}{5}\right] = 2$$

The jump occurs at the next multiple of 5.

$$f(15,5) = \left[\frac{15}{5}\right] = 3.$$

Now set $p = 2$. Then the jumps are more frequent, and a big jump occurs when $n = p^k$, for some $k = 0, 1, 2, \ldots$.

$$f(1,2) = 0,\ f(2,2) = 1,\ f(3,2) = 1$$

$$f(4,2) = \left[\frac{4}{2}\right] + \left[\frac{4}{2^2}\right] = 2 + 1 = 3\ldots$$

$$f(8,2) = \left[\frac{8}{2}\right] + \left[\frac{8}{2^2}\right] + \left[\frac{8}{2^3}\right] = 4 + 2 + 1 = 7\ldots$$

$$f(10,2) = \left[\frac{10}{2}\right] + \left[\frac{10}{2^2}\right] + \left[\frac{10}{2^3}\right] + \ldots = 5 + 2 + 1 = 8$$

$$f(11,2) = \left[\frac{11}{2}\right] + \left[\frac{11}{2^2}\right] + \left[\frac{11}{2^3}\right] + \ldots = 5 + 2 + 1 = 8$$

$$f(12,2) = \left[\frac{12}{2}\right] + \left[\frac{12}{2^2}\right] + \left[\frac{12}{2^3}\right] + \ldots = 6 + 3 + 1 = 10$$

$$f(13,2) = \left[\frac{13}{2}\right] + \left[\frac{13}{2^2}\right] + \left[\frac{13}{2^3}\right] + \ldots = 6 + 3 + 1 = 10$$

$$f(14,2) = \left[\frac{14}{2}\right] + \left[\frac{14}{2^2}\right] + \left[\frac{14}{2^3}\right] + \ldots = 7 + 3 + 1 = 11$$

$$f(15,2) = \left[\frac{15}{2}\right] + \left[\frac{15}{2^2}\right] + \left[\frac{15}{2^3}\right] + \ldots = 7 + 3 + 1 = 11$$

$$f(16,2) = \left[\frac{16}{2}\right] + \left[\frac{16}{2^2}\right] + \left[\frac{16}{2^3}\right] + \left[\frac{16}{2^4}\right] \ldots = 8 + 4 + 2 + 1 = 15$$

Note that $f(n,2)$ exceeds $f(n,5)$. Collecting the results for $n = 1, 2, \ldots 15$ we find that $1!, 2!, \ldots 4!$ are not divisible by 10,

$$\begin{aligned} 5! &= 120 \\ 6! &= 720 \\ 7! &= 5040 \\ 8! &= 40320 \\ 9! &= 362,880 \end{aligned}$$

are divisible by 10 and have one zero at the end.

$$\begin{aligned} 10! &= 2^8 5^2 \ldots = 3,628,800 \\ 11! &= 2^8 5^2 \ldots = 39,916,800 \\ 12! &= 2^{10} 5^2 \ldots = 479,001,600 \\ 13! &= 2^{10} 5^2 \ldots = 6,227,020,800 \\ 14! &= 2^{11} 5^2 \ldots = 87,178,291,200 \\ 15! &= 2^{11} 5^3 \ldots = 1,307,674,368,000 \end{aligned}$$

are divisible by 10^2, hence ending in two zeros whereas 15! is divisible by 10^3, and ends in three zeros. The pattern continues $15!, \ldots 19!$ are divisible by 10^3 etc. So when the exponent of 5 is large enough in $n!$ to produce 99 zeros then four more consecutive numbers $n+1, n+2, n+3, n+4$ will produce the same number of zeros at the end of their factorials.

9.4 Solutions 56-67.

- Mathematics Test (GRE)
- *Spartan Old School Tutorials*
- Last revision August 1, 2015

Problems

56.

$$\omega(x, y) = (x - y)^2$$

is not a metric on the set of real numbers. Set $x = 1$, $y = 2$, $z = 3$ then

$$\omega(x, y) + \omega(y, z) < \omega(x, z).$$

violates the *triangle inequality.*

57. The set of real numbers for which the series

$$\sum_{n=1}^{\infty} \frac{n! x^{2n}}{n^n (1 + x^{2n})}$$

converges is R.

Solution: Write

$$a_n = \frac{n!}{n^n}, \ b_n = \frac{n! x^{2n}}{n^n (1 + x^{2n})}.$$

$$0 \leq b_n \leq a_n.$$

because

$$0 \leq \frac{x^{2n}}{(1 + x^{2n})} < 1.$$

By the *Comparison Test,* it is enough to show that $\sum_{n=1}^{\infty} a_n$ converges. Apply the *d'Alambert or Quotient Test:*

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{(n+1)}} \frac{n^n}{n!} = \left(\frac{n}{n+1} \right)^n = \left(1 - \frac{1}{n+1} \right)^n.$$

$$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = \exp(-1), \text{monotone decreasing}$$

When $n > 1$

$$\left(1 - \frac{1}{n+1}\right)^n \leq 0.5 < 1.$$

Thus $\sum_{n=1}^{\infty} a_n$ converges, hence $\sum_{n=1}^{\infty} b_n$ does, too.

58. Suppose A and B are $n \times n$ invertible matrices, where $n > 1$ and I is the $n \times n$ identity. If A and B are similar matrices then

(I) $A - 2I$ and $B - 2I$ are similar matrices.
(II) A and B have the same trace.
(III) A^{-1} and B^{-1} are similar.

Proof: A, B are similar if there exists nonsingular P such that

$$A = PBP^{-1}.$$

Write

$$A \approx B$$

for similar matrices.

$$P(B - 2I)P^{-1} = PBP^{-1} - 2PIP^{-1} = PBP^{-1} - 2I(PP^{-1}) = A - 2I$$

proves (I).

If $A \approx B$ then

$$\det A = (\det P)(\det B)(\det P^{-1}) = \det B$$

because $(\det P)(\det P^{-1}) = 1$. $A - \lambda I$ and $B - \lambda I$ are similar matrices (see (I)) so their determinants are equal. But the determinants are polynomials in λ, actually, they are the *characteristic polynomials* of A and B, respectively, and they coincide. So there is one and the same characteristic polynomial for A and B:

$$c(\lambda) = \sum_{i=0}^{n} c_i \lambda_i$$

$$\text{tr}A = \text{tr}B = -c_{n-1} = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} b_{ii}.$$

This concudes the proof of claim (II). As for claim(III)

$$A^{-1} = (PBP^{-1})^{-1} = (P^{-1})^{-1}B^{-1}P^{-1} = PB^{-1}P^{-1}.$$

59. Let f be an analytic function of the complex variable $z = x + \imath y$ given by

$$f(z) = (2x + 3y) + \imath g(x, y)$$

where $g(x, y)$ is a real-valued function of the real variables x and y. If $g(2, 3) = 1$, then $g(7, 3) =$.

Proof: Change notation:

$$f(z) = u(x, y) + \imath v(x, y)$$

$$u(x, y) = 2x + 3y$$

$$v(x, y) = g(x, y)$$

Cauchy-Riemann differential equations are satisfied by analytic functions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 3$$

$$\int \frac{\partial v}{\partial y} dy = \int 2dy = 2y + h(x)$$

$$-\frac{\partial v}{\partial x} = -h'(x) = 3$$

$$h(x) = -3x + c$$

$$v(x, y) = 2y - 3x + c$$

$$v(2, 3) = 2 * 3 - 3 * 2 + c = 1 \;\Rightarrow\; c = 1$$

$$v(7, 3) = 2 * 3 - 3 * 7 + 1 = -14.$$

60. The group of symmetries of the regular pentagram is isomorphic to the *dihedral group of order* 10.

Proof: Let P_0, P_1, P_2, P_3, P_4 be the vertices of the regular pentagram in a clockwise order. Let C be the centre of the convex hull of the regular pentagram (which is the regular pentagon). Write R for a rotation of the regular pentagram (as well as its convex hull) by $\frac{2\pi}{5}$ in its plane about C. This rotation has order 5. Write D for the reflection of the regular pentagram (as well as its convex hull) in the axis through P_0 and C. D has order 2. They generate $2 * 5 = 10$ different symmetries:

$$\{1, R, R^2, R^3, R^4, D, DR, DR^2, DR^3, DR^4\}$$

This is the complete list of symmetries for the regular pentagram (as well as its convex hull); $R's$ keep the cyclic order of vertices, D reverses it. The above group is Δ_5, the dihedral group of the regular pentagon.

61. Cardinal numbers for certain sets.

Reminder: The cardinal number (or cardinality) is the property that sets A and B have in common if they can be put into a one-to-one correspondence. The cardinal number for a finite set is the (counting) number of its elements, a natural number. The smallest *infinite cardinal* is $\aleph_0$. The set of natural numbers $N = \{0, 1, 2, 3, \ldots\}$ and all sets that have bijection with it are *countably infinite* and have the cardinal number $\aleph_0$.

$$|A| = |B| = \aleph_0$$

(we use the $|.|$ notation for cardinal numbers of sets in this paragraph.)

The set of real numbers is not countable, in fact, even the set S of real numbers between 0 and 1 is not countable (see Cantor's classic theorem). The cardinal number for S is c, often called the cardinal number of continuum; or the cardinal number for *uncountably infinite* sets. It is the second infinite cardinal number.

Further, it can be shown that for any set A there does not exist a function mapping A onto its power set $P(A)$, the set of all subsets of A, and

$$|P(A)| > |A|.$$

The set of characteristic functions on A that is functions taking on only the values of 0 and 1 are equvivalent to the subsets of A therefore

$$|P(A)| = 2^{|A|}.$$

As suggested above, cardinal numbers form a chain under " $\leq$ ", and the first cardinals are :

$$n \leq \aleph_0 \leq c \leq 2^c.$$

Next, we quote some definitions *(Kaplansky)*. Let d and e be cardinal numbers. Take the disjoint sets D and E with $d = |D|$ and $e = |E|$. Then $d + e = |D \cup E|$. If $d \leq e$ and e is infinite then $d + e = e$.

For their product, $de = |D \times E|$. Further, if $d \neq 0$, $d \leq e$, e is infinite then $de = e$.

If e and d are nonzero cardinal numbers, then we define d^e to be the cardinal number of the set of all functions from E to D.

Cardinal numbers for certain sets:

1) $|R| = c$

2) | The set of all functions from Z to $Z| = \aleph^{\aleph} = c$.

3) | The set of all functions from R to $\{0,1\}| = 2^c$.

4) | The set of all finite subsets of R $| = c$.

5) | The set of all polynomials with coefficients in R $| = c$.

62. Let K be a nonempty subset of R^n, where $n > 1$.

(I) If K is compact, then every continuous real-valued function defined on K is bounded.

(II) If every every continuous real-valued function defined on K is bounded, then K is compact.

Proof: (I) If K is compact then K is closed and bounded. Then for a continuous function f on K there exist points P and Q in K such that

$$f(P) \leq f(x) \leq f(Q), \quad \forall x \epsilon K$$

and $f(x)$ attains its minimum $f(P)$ and maximum $f(Q)$ on K. Evidently $f(x)$ is bounded by these values.

(II) Contraposition: If K is noncompact then there exists a continuous function on K which is not bounded. Let K be R^2 with $P = P(x_0, y_0)$ removed, that is the pierced plane. Set

$$f(x,y) = \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2)}}.$$

K is noncompact, $f(x,y)$ is continuous and unbounded on K.

63. If f is the function defined by

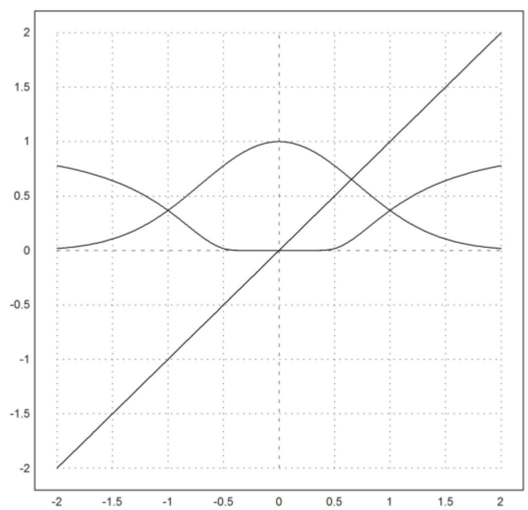
$$f(x) = \begin{cases} x\exp(-x^2)\exp(-x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

at how many values of x does the graph of f have horizontal tangent line?

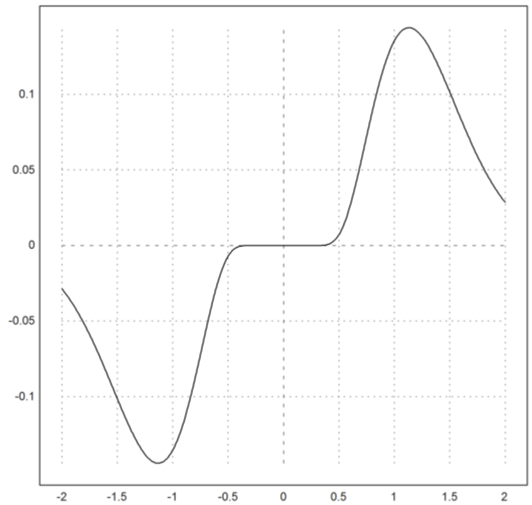
Inspection: Write

$$\begin{aligned} f_1(x) &= x \\ f_2(x) &= \exp(-x^2) \\ f_3(x) &= \exp(-x^{-2}) \\ f(x) &= f_1(x)f_2(x)f_3(x). \end{aligned}$$

Upon displaying the component functions, we recognize $f_1(x), f_2(x)$ easily, perhaps $f_3(x)$ is less familiar.



This suggests that $f(x)$ has horizontal tangents at three values.



Solution: We want to find the values of x for which $f'(x) = 0$.

$$\begin{aligned} f_1'(x) &= 1 \\ f_2'(x) &= -2xf_2(x) \\ f_3'(x) &= 2x^{-3}f_3(x) \\ f'(x) &= f_1'(x)f_2(x)f_3(x) + f_1(x)f_2'(x)f_3(x) + f_1(x)f_2(x)f_3'(x) \\ &= f_2(x)f_3(x) + f_1(x)(-2x)f_2(x)f_3(x) + f_1(x)f_2(x)(2x^{-3})f_3(x) \\ &= f_2(x)f_3(x) - 2x^2f_2(x)f_3(x) + 2x^{-2}f_2(x)f_3(x) \\ &= (1 - 2x^2 + 2x^{-2})f_2(x)f_3(x). \end{aligned}$$

$$1 - 2x^2 + 2x^{-2} = 0$$

$$x^2 - 2x^4 + 2 = 0; \; \xi = x^2$$

$$-2\xi^2 + \xi + 2 = 0$$

$$\xi_{1,2} = \frac{-1 \pm \sqrt{17}}{-4}; \; \xi_1 = \frac{-1 - 4.1231}{-4} = 1.2807, \; x_{1,2} = \pm 1.1317$$

Therefore $f(x)$ has horizontal tangent at $x_{1,2} = \pm 1.1317$ Further, there is a horizontal tangent at $x_0 = 0$

$$\lim_{\Delta x \to 0} \frac{f(\Delta x) - 0}{\Delta x} = f_2(\Delta x)f_3(\Delta x) = 0$$

$$\lim_{-\Delta x \to 0} \frac{f(-\Delta x) - 0}{-\Delta x} = f_2(-\Delta x) f_3(-\Delta x) = 0.$$

64. For each positive n, let f_n be the function defined on the interval $[0, 1]$ by $f_n(x) = \dfrac{x^n}{1 + x^n}$. Prove the following statements:

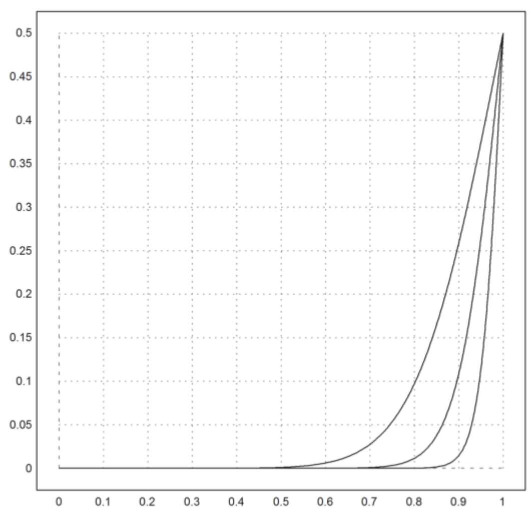
I. The sequence $f_n(x)$ converges pointwise on $[0, 1]$ to a limit function f.

II. The sequence $f_n(x)$ does not converge uniformly on $[0, 1]$ to a limit function f.

III. Integral and limit commute.

Inspection: Graphs of f_{10}, f_{20}, f_{40} show

$$f_{10}(x) \geq f_{20} \geq f_{40}; \;\; x \epsilon [0, 1]$$



$$f_{10}(0) = f_{20}(0) = f_{40}(0) = 0$$

$$f_{10}(1) = f_{20}(1) = f_{40}(1) = 0.5,$$

as n increases the corner around $x \approx 1$ gets sharper and sharper, the limit function looks discontinuous, and

$$\int_0^1 f_n(x) dx \Rightarrow 0.$$

Proof: (I):

$$\lim_{n\to\infty} \frac{x^n}{1+x^n} = 0,\ 0 \le x < 1$$

$$\lim_{n\to\infty} \frac{x^n}{1+x^n} = 0.5,\ x = 1.$$

$$\lim_{n\to\infty} f_n(x) = f(x),$$

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 0.5 & x = 1 \end{cases}$$

(II): Suppose - if possible - that $f_n(x)$ converges uniformly on $[0,1]$. Then for every $0 < \epsilon \ll 1$ there exists an N such that

$$|f_n(x) - f_m(x)| < \epsilon,\ n,m \ge N,\ x\epsilon[0,1].$$

Let ϵ be a small positive number, fixed. Then there is a N_0 that satisfies the above equation. Since $f_n(x)$ is a continuous, positive, monotone increasing function that takes on all values of $[0,0.5]$, $f_n(0) = 0,\ f_n(1) = 0.5$, there is an n, and x_0 such that

$$f_n(x_0) = 2\epsilon,\ 0 < x_0 < 1; n > N_0.$$

Clearly, $x_0 \ne 1$, $x_0 \ne 0$. For this x_0 we can have $m > n > N_0$ such that

$$0 < f_m(x_0) \le \epsilon.$$

$$|f_n(x_0) - f_m(x_0)| > \epsilon.$$

This contradicts the assumption, therefore $f_n(x)$ does not converge uniformly on $[0,1]$.

(III):

$$0 < \eta << 1$$

$$\int_0^1 \frac{x^n}{1+x^n}dx = \int_0^1 x\frac{x^{n-1}}{1+x^n}dx = \xi\int_0^1 \frac{x^{n-1}}{1+x^n}dx,\quad \xi\epsilon[0,1]$$

by one of the Mean Value Theorems on integrals. Then for large n

$$\int_0^1 \frac{x^n}{1+x^n}dx \leq \int_0^1 \frac{x^{n-1}}{1+x^n}dx = \left[\frac{1}{n}\ln(1+x^n)\right]_0^1 = \frac{\ln(2)}{n} < \eta.$$

$$\lim_{n \to \infty} \int_0^1 \frac{x^n}{1+x^n}dx = 0.$$

From the definition of $f(x)$ in (I)

$$\int_0^1 \lim_{n \to \infty} \frac{x^n}{1+x^n}dx = \int_0^1 f(x)dx = 0.$$

$$\lim_{n \to \infty} \int_0^1 \frac{x^n}{1+x^n}dx = \int_0^1 \lim_{n \to \infty} \frac{x^n}{1+x^n}dx.$$

65. There is a continuous function from $(0,1)$ onto $[0,1]$. Take $f(x) = 0.5\sin(2\pi x) + 0.5$ defined on $(0,1)$.

66. Let R be a ring with multiplicative identity. If U is an additive subgroup of R such that $ur \epsilon U$ for all $u \epsilon U$ and for all $r \epsilon R$, then U is said to be a right ideal of R. If R has exactly two right ideals, then R is a division ring.

Remarks: First, a *division ring* is a nontrivial ring, not necessarily commutative, in which every non-zero element has two-sided multiplicative inverse. A noncommutative division ring is called a *skew field* by some algebraists, and a commutative division ring is a *field* .

Second, two ideals of a ring R are R itself and the trivial ideal (denoted by 0), which has only the zero element.

Third, an element x in a ring R with identity 1 is said to be *left invertible* if there exists x_L^{-1} such that

$$x_L^{-1}x = 1.$$

Similar definition for *right invertible* :

$$xx_R^{-1} = 1.$$

If both right and left inverses exist then they coincide.

$$x_L^{-1}x = 1 = xx_R^{-1}$$

$$x_R^{-1} = 1x_R^{-1} = (x_L^{-1}x)x_R^{-1} = x_L^{-1}(xx_R^{-1}) = x_L^{-1}1 = x_L^{-1}.$$

Fourth, let us examine the question of one-sided inverses in a ring with identity. Suppose in ring R every non-zero element has a one-sided multiplicative inverse. Write a for a typical element, then there exist b such that

$$ab = 1.$$

But b too, has a one-sided multiplicative inverse c

$$bc = 1.$$

Therefore

$$abc = c$$

$$a(bc) = c$$

$$a = c$$

and

$$ab = ba = 1.$$

This means that a and b are each other's two-sided inverses. Serendipity. Fifth, the most interesting example of division rings is the *ring of quaternions.* The quaternions are a four-dimensional vector space over real numbers with basis $1, \imath, j, k$. A typical quaternion is

$$a + b\imath + cj + dk$$

where a, b, c, d are real numbers. If $c = d = 0$ then the resulting subfield is isomorphic to the field of complex numbers. The sum of quaternions is obvious, the product is determined by the ring axioms and the rules

$$\imath^2 = j^2 = k^2 = -1; \quad ij = k = -ji; \quad jk = i = -kj; \quad ki = j = -ik.$$

These rules were carved in Brougham Bridge by W. R. Hamilton in 1843 A.D. There is an alternative representation by special 2×2 matrices. The quaternions are discussed in every good algebra textbooks.

Proof: Let $x \neq 0$ be an element of R. The right ideal generated by this single element is a *principle ideal* and is denoted by (x). Since R has exactly two right ideals, (x) is either the trivial ideal or R itself; but $x \epsilon (x)$ indicates $(x) \neq 0$, and the principal ideal generated by $x \neq 0$ is R, the ring itself. Therefore $x \neq 0$ has a right inverse, and by the remarks above it has a two-sided multiplicative inverse.

67.