Pinter Consulting Quarterly Reports New Series No. 5.

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September 30, 2014

Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

Előszó

Ült Dr No egyedűl zöld sátorboltnak alatta.
Mély vágás nem volt testén; de körűl vala sebbel
Dárdacsapás sajgott fájdalmasan égve bokáin;
Combja szurást vett, és keze, melle, bal oldala vérzett;
Vére lefolyt testén, és a por alatta megázék.

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Introduction

Pinter Consulting of Calgary, Alberta practices Mathematics, promotes clear thinking and offers Consultations, Tutorials and Seminars in Mathematics.

Contents

Chapter 5

Proceedings

5.1 Summary of Current Report

- **Private study for professional development:**
- Records of activities at Pinter Consulting
- Collection of problems with our own solutions .
- Socratic Programme
  - Analysis:
  - Algebra and Number Theory:
  - Geometry:
  - Differential and Integral Equations:
- Continuos improvement, corrections and last revisionNovember 6, 2014.

5.2 Tutorial 1.

- Analysis
- *Spartan Old School* Δ
- Last revision September 30, 2014

Exercises

46.

$$\lim_{n \to \infty} \frac{10000n}{n^2+1} = 0.$$

Proof:

$$0 \leq \frac{10000n}{n^2+1} = \frac{10000}{n + \dfrac{1}{n}} \leq \frac{10000}{n}$$

$$0 \leq \lim_{n \to \infty} \frac{10000n}{n^2+1} \leq \lim_{n \to \infty} \frac{10000}{n} = 0.$$

47.

$$\lim_{n \to \infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0.$$

Proof:

$$\left(\sqrt{n+1} - \sqrt{n}\right) = \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)} = \frac{1}{\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$\frac{1}{2\sqrt{n+1}} < \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$$

$$\lim_{n \to \infty} \frac{1}{2\sqrt{n+1}} = \lim_{n \to \infty} \frac{1}{2\sqrt{n}} = 0.$$

49.

$$\lim_{n \to \infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \frac{1}{3}.$$

$$\frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \frac{\left[\left(\frac{-2}{3}\right)^n + 1\right] 3^n}{\left[\left(\frac{-2}{3}\right)^{n+1} + 1\right] 3^{n+1}} = \frac{\left[\left(\frac{-2}{3}\right)^n + 1\right]}{\left[\left(\frac{-2}{3}\right)^{n+1} + 1\right]} \times \frac{1}{3}$$

$$\lim_{n \to \infty} \frac{\left[\left(\frac{-2}{3}\right)^n + 1\right]}{\left[\left(\frac{-2}{3}\right)^{n+1} + 1\right]} = 1.$$

48.

$$\lim_{n \to \infty} \frac{n^{\frac{2}{3}} \sin(n!)}{n+1} = 0$$

Proof:

$$|\sin(n!)| \leq 1$$

$$\left|\frac{n^{\frac{2}{3}} \sin(n!)}{n+1}\right| \leq \left|\frac{n^{\frac{2}{3}}}{n+1}\right| |\sin(n!)| \leq \left|\frac{n^{\frac{2}{3}}}{n+1}\right| \leq \left|\frac{n^{\frac{-1}{3}}}{1+n^{-1}}\right| \leq \left|n^{\frac{-1}{3}}\right|$$

$$\lim_{n \to \infty} \left|n^{\frac{-1}{3}}\right| = 0.$$

50.

$$\lim_{n \to \infty} \frac{1 + a + a^2 + a^3 + \ldots + a^n}{1 + b + b^2 + b^3 + \ldots + b^n} = \frac{1-b}{1-a}; \ (|a| < 1, |b| < 1)$$

Proof:

$$1 + a + a^2 + a^3 + \ldots + a^n = \frac{a^{n+1} - 1}{a - 1} = \frac{1 - a^{n+1}}{1 - a} \Longrightarrow \frac{1}{1-a}; \ n \to \infty$$

$$1 + b + b^2 + b^3 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1} = \frac{1 - b^{n+1}}{1 - b} \Longrightarrow \frac{1}{1-b}; \ n \to \infty$$

$$\lim_{n \to \infty} \frac{1 - a^{n+1}}{1 - a} \frac{1 - b}{1 - b^{n+1}} = \frac{1-b}{1-a}.$$

51.

$$\lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n-1}{n^2}\right) = \frac{1}{2}$$

Proof:

$$1 + 2 + 3 + \ldots + (n-1) = \frac{(n-1)n}{2} = \frac{n^2 - n}{2}$$

$$\lim_{n \to \infty} \frac{n^2 - n}{2n^2} = \lim_{n \to \infty} \frac{1 - \frac{1}{n}}{2} = \frac{1}{2}.$$

52.

$$\lim_{n \to \infty} \left|\frac{1}{n} - \frac{2}{n} + \frac{3}{n} - \ldots + \frac{(-1)^{n-1}n}{n}\right| = \frac{1}{2}$$

Proof:

$$\begin{aligned} 1 - 2 + 3 &= +2 \\ 1 - 2 + 3 - 4 &= -2 \\ 1 - 2 + 3 - 4 + 5 &= +3 \\ 1 - 2 + 3 - 4 + 5 - 6 &= -3 \\ &\vdots \\ 1 - 2 + 3 - 4 + 5 + \ldots + (2n-1) &= +n \\ 1 - 2 + 3 - 4 + 5 + \ldots\ldots\ldots - 2n &= -n \end{aligned}$$

$$\frac{1}{n} - \frac{2}{n} + \frac{3}{n} - \ldots + \frac{(-1)^{n-1}n}{n} = \begin{cases} \dfrac{n}{2n-1} \\ \dfrac{-n}{2n} \end{cases}$$

$$\left|\frac{n}{2n-1}\right| \Longrightarrow \frac{1}{2}; \quad \left|\frac{-n}{2n}\right| \Longrightarrow \frac{1}{2}; \quad n \to \infty.$$

53.

$$\lim_{n \to \infty} \left[\frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \ldots + \frac{(n-1)^2}{n^3}\right] = \frac{1}{3}.$$

Proof:

$$1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 = \frac{(n-1)(n)(2n-1)}{6} = \frac{2n^3 + O(n^2)}{6}$$

$$\lim_{n \to \infty} \frac{2n^3 + O(n^2)}{6n^3} = \frac{2}{6} = \frac{1}{3}.$$

54.

$$\lim_{n \to \infty} \left[\frac{1^2}{n^3} + \frac{3^2}{n^3} + \frac{5^2}{n^3} + \ldots + \frac{(2n-1)^2}{n^3}\right] = \frac{4}{3}.$$

Proof:

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3} = \frac{4n^3 + O(n)}{3}$$

$$\lim_{n \to \infty} \frac{4n^3 + O(n)}{3n^3} = \frac{4}{3}.$$

Remarks:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^2 + 3^2 + 5^2 \ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

For proofs by induction see *Tutorials on Higher Arithmetic.*

55.

$$\lim_{n \to \infty} \left(\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n}\right) = 3.$$

Proof: Consider the following triangular arrangement:

$$\begin{array}{cccccccc}
\frac{1}{2} & & & & & & & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & & & & \frac{3}{2^2} \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & & & \frac{5}{2^3} \\
\frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{7}{2^4} \\
\ldots & \ldots & \ldots & \ldots & & & & \ldots
\end{array}$$

First column

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1.$$

Second and third

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{1}{2}; \ \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{1}{2}.$$

Fourth and fifth

$$\frac{1}{8} + \frac{1}{16} + \ldots = \frac{1}{4}; \ \frac{1}{8} + \frac{1}{16} + \ldots = \frac{1}{4} \quad \text{etc.}$$

$$1 + (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8}) + \ldots = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 3.$$

56.

$$\lim_{n \to \infty} \left(\frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \ldots + \frac{1}{n*(n+1)}\right) = 1.$$

Proof:

$$\begin{aligned}
a_n &= \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \ldots + \frac{1}{n*(n+1)} \\
&= \frac{2-1}{1*2} + \frac{3-2}{2*3} + \frac{4-3}{3*4} + \ldots + \frac{(n+1)-n}{n*(n+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{2}{1*2} - \frac{1}{1*2}\right) + \left(\frac{3}{2*3} - \frac{2}{2*3}\right) + \left(\frac{4}{3*4} - \frac{3}{3*4}\right) + \ldots \\
&+ \left(\frac{n+1}{n*(n+1)} - \frac{n}{n*(n+1)}\right) \\
&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \ldots + \frac{1}{n} - \frac{1}{n+1} \\
&= 1 - \frac{1}{n+1}.
\end{aligned}
$$

$$\lim_{n \to \infty} a_n = 1.$$

57.

$$\lim_{n \to \infty} \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\ldots\sqrt[2n]{2} = 2.$$

Proof:

$$
\begin{aligned}
a_n &= \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\ldots\sqrt[2n]{2} \\
\ln a_n &= \frac{1}{2}\ln 2 + \frac{1}{4}\ln 2 + \frac{1}{8}\ln 2 + \ldots + \frac{1}{2^n}\ln 2 \\
&= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}\right)\ln 2
\end{aligned}
$$

$$\lim_{n \to \infty} \ln a_n = \ln 2.$$

58.

$$\lim_{n \to \infty} \frac{n}{2^n} = 0.$$

Proof:

$$
\begin{aligned}
n &> 1 \\
\frac{n}{2^{(n+1)}} &> \frac{1}{2^{(n+1)}} \\
\frac{n}{2^{(n+1)}} + \frac{n}{2^{(n+1)}} &> \frac{n}{2^{(n+1)}} + \frac{1}{2^{(n+1)}} \\
\frac{n}{2^n} &> \frac{n+1}{2^{n+1}}
\end{aligned}
$$

Therefore $\frac{n}{2^n}$ is strictly monotone decreasing. Further, let us examine

$$2^k > 2k; \quad k > 2.$$

The inequality holds for $k = 3$ $(8 > 4)$. Suppose it is true for some $k \geq 3$, then since

$$2 > \frac{k+1}{k}, \quad \forall k > 2$$

$$2 * 2^k > 2k * \frac{k+1}{k},$$

$$2^{k+1} > 2 * (k+1).$$

Therefore the inequality is true for all $k > 3$. Now let us choose a small positve number $\varepsilon < \frac{1}{2^3}$. Then $\exists k$ such that $\frac{1}{2^k} \geq \varepsilon > \frac{1}{2^{k+1}}$. Write $N = N(\varepsilon) = 2^{k+1}$. Next, we show that $\frac{N}{2^N} < \varepsilon$ by the above inequality. Indeed,

$$\frac{N}{2^N} \leq \frac{2^{k+1}}{2^{2^{k+1}}} < \frac{2^{k+1}}{2^{2*(k+1)}} < \frac{2^{k+1}}{2^{(k+1)+(k+1)}} = \frac{1}{2^{(k+1)}} < \varepsilon.$$

Since $\frac{n}{2^n}$ is strictly monotone decreasing and positive we have

$$0 < \frac{n}{2^n} < \varepsilon; \ \forall n, \ n > N(\varepsilon).$$

Proof is complete: $\frac{n}{2^n}$ converges to 0 as $n \to \infty$.

5.3 Tutorial 2.

- Higher Arithmetic
- *Spartan Old School* Γ
- Last revision September 30, 2014

Exercises

1.

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Proof:

$$1 = \frac{1(1+1)}{2} = 1.\surd$$

$$1 + 2 = \frac{2(2+1)}{2} = 3.\surd$$

$$1 + 2 + 3 = \frac{3(3+1)}{2} = 6.\surd$$

Induction hypothesis:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

$$1 + 2 + 3 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) =$$

$$\frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}.\surd$$

2.

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^2 = \frac{1(1+1)(2+1)}{6} = 1. \surd$$

$$1^2 + 2^2 = \frac{2(2+1)(4+1)}{6} = 5. \surd$$

$$1^2 + 2^2 + 3^2 = \frac{3(3+1)(6+1)}{6} = 14. \surd$$

Induction hypothesis:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$$

$$\frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} = \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} =$$

$$\frac{(n+1)[2n^2 + n + 6n + 6]}{6} = \frac{(n+1)[2n^2 + 7n + 6]}{6} = \frac{(n+1)(n+2)(2n+3)}{6}. \surd$$

3.

$$(1^3 + 2^3 + 3^3 + \ldots + n^3) = (1 + 2 + 3 + \ldots + n)^2$$

Proof (sketch):

$$(1^3 + 2^3 + 3^3 + \ldots + n^3) = \left(\frac{n(n+1)}{2}\right)^2$$

$$(1^3 + 2^3 + 3^3 + \ldots + n^3) + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 =$$

$$\frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2}{4}\left(n^2 + 4n + 4\right) = \frac{(n+1)^2}{4}(n+2)^2 =$$

$$\left(\frac{(n+1)(n+2)}{2}\right)^2. \surd$$

4.

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1.$$

Proof (direct):

$$S_0 = 1 = 2^1 - 1 = 1.\surd$$

$$S_1 = 1 + 2 = 2^2 - 1 = 3.\surd$$

$$S_2 = 1 + 2 + 4 = 2^3 - 1 = 7.\surd$$

$$S_{n-1} = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1$$

$$S_n = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} + 2^n$$

$$S_n - 2 * S_{n-1} = 1;$$

$$S_n = 2 * S_{n-1} + 1 = 2 * (2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1.\surd$$

5. The *Fibonacci numbers* are defined by

$$F_1 = F_2 = 1; F_n = F_{n-1} + F_{n-2},\ n > 2.$$

The first Fibonacci numbers are:

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

Set $\tau = \frac{1+\sqrt{5}}{2}$, ($\tau$ is called the *golden ratio*) and $\sigma = \frac{-1}{\tau}$. Show that

$i)\ F_n < \tau^n$

$ii)\ F_n = \frac{(\tau^n - \sigma^n)}{\sqrt{5}}$

Remark: The *golden ratio* originates in the proportion

$$1 : x = x : (x + 1).$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \tau,\ \sigma;$$

for

$$\sigma = -\frac{1}{\tau} = -\frac{2}{1+\sqrt{5}} = -\frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = -\frac{2(1-\sqrt{5})}{1-5} = \frac{1-\sqrt{5}}{2}.$$

Proof:

$$\tau = \frac{1+\sqrt{5}}{2};\ 3 > \sqrt{5} > 2$$

$$\tau^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + \tau$$

$$\tau^3 = \tau\tau^2 = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right) = 2 + \sqrt{5}.$$

Therefore

$$F_1 < \tau;\ F_2 < \tau^2;\ F_3 < \tau^3.$$

Hypothesis

$$F_{n-2} < \tau^{n-2};\ F_{n-1} < \tau^{n-1}; n \geq 3.$$

Induction

$$F_{n-2} + F_{n-1} < \tau^{n-2} + \tau^{n-1} = \tau^{n-2}(1+\tau) = \tau^{n-2}\left(1 + \frac{1+\sqrt{5}}{2}\right) =$$

$$\tau^{n-2}\left(\frac{3+\sqrt{5}}{2}\right) = \tau^{n-2}\tau^2 = \tau^n.$$

Thus by the recursive definition of Fibonacci numbers

$$F_{n-2} + F_{n-1} = F_n < \tau^n,$$

and Claim i) is proven.

Next, we show

$$F_n = \frac{(\tau^n - \sigma^n)}{\sqrt{5}}; \; n = 1, 2.$$

$$F_1 = \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)\left(\sqrt{5}\right)^{-1} = 1.\surd$$

$$F_2 = \left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right)\left(\sqrt{5}\right)^{-1} =$$

$$= \left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right)\left(\left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)\right)\left(\sqrt{5}\right)^{-1} =$$

$$= \left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}+1-\sqrt{5}}{2}\right)\left(\sqrt{5}\right)^{-1} =$$

$$= \left(\frac{\sqrt{5}+\sqrt{5}}{2}\right)\left(\frac{1+1}{2}\right)\left(\sqrt{5}\right)^{-1} = \left(\sqrt{5}\right)\left(\sqrt{5}\right)^{-1} = 1.\surd$$

Moreover,

$$\sigma^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = 1+\sigma,$$

and recall

$$\tau^2 = 1 + \tau.$$

Hence

$$F_n = \frac{(\tau^n - \sigma^n)}{\sqrt{5}} = \frac{(\tau^2\tau^{n-2} - \sigma^2\sigma^{n-2})}{\sqrt{5}} = \frac{(\tau+1)\tau^{n-2} - (\sigma+1)\sigma^{n-2}}{\sqrt{5}}$$

$$= \frac{\tau^{n-1} - \sigma^{n-1}}{\sqrt{5}} + \frac{\tau^{n-2} - \sigma^{n-2}}{\sqrt{5}} = F_{n-1} + F_{n-2},$$

which proves Claim ii) for $n \geq 3$.

1.06. *Express* 22! *as a product of prime factors.*

Let p be a prime number. Then the exact power of p that divides $n!$ is given by

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots$$

where there are only finite number of non-zero terms in the series. Then

$$n! = 1 \cdot 2 \cdots (p-1) \cdot p \cdot (p+1) \cdots p^2 \cdots$$

and there are $\left[\frac{n}{p}\right]$ that are divisible by p, out of these there are $\left[\frac{n}{p^2}\right]$ divisible by p^2; moreover, the multiples of p^3 number $\left[\frac{n}{p^3}\right]$ etc. We continue until we reach p^m where m is the highest power of p contained in n. Take $p = 2$,

$$2^4 < 22 < 2^5$$

$$\left[\frac{22}{2}\right] + \left[\frac{22}{2^2}\right] + \left[\frac{22}{2^3}\right] + \left[\frac{22}{2^4}\right] + \left[\frac{22}{2^5}\right] \dots =$$

$$[11] + [5.5] + [2.75] + [1.375] + [0.6875] =$$

$$11 + 5 + 2 + 1 + 0 = 19.$$

Next, $p = 3$

$$3^2 < 22 < 3^3$$

$$\left[\frac{22}{3}\right] + \left[\frac{22}{3^2}\right] + \left[\frac{22}{3^3}\right] + \dots =$$

$$[7.3333] + [2.4444] + [0.8148] = 7 + 2 + 0 = 9.$$

For the other primes $p = 5, 7, 11, 13, 17, 19$, less than 22 we have

$$\left[\frac{22}{5}\right] = 4; \quad \left[\frac{22}{5^2}\right] = \left[\frac{22}{5^3}\right] = \dots = 0$$

$$\left[\frac{22}{7}\right] = 3; \quad \left[\frac{22}{7^2}\right] = \left[\frac{22}{7^3}\right] = \dots = 0$$

$$\left[\frac{22}{11}\right] = 2; \quad \left[\frac{22}{11^2}\right] = \left[\frac{22}{11^3}\right] = \ldots = 0$$

$$\left[\frac{22}{13}\right] = \left[\frac{22}{17}\right] = \left[\frac{22}{19}\right] = 1$$

$$\left[\frac{22}{13^2}\right] = \left[\frac{22}{17^2}\right] = \left[\frac{22}{19^2}\right] = 0$$

Therefore

$$22! = 2^{19} \times 3^9 \times 5^4 \times 7^3 \times 11^2 \times 13 \times 17 \times 19.$$

Check:

$$\begin{aligned}
1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 &\quad \times \\
12 \times 13 \times 14 \times 15 \times 16 \times 17 \times 18 \times 19 \times 20 \times 21 \times 22 &\quad = \\
2^{19} \times 3^9 \times 5^4 \times 7^3 \times 11^2 \times 13 \times 17 \times 19. &
\end{aligned}$$

$$\begin{aligned}
1 \times 2 \times 3 \times 2^2 \times 5 \times (2 * 3) \times 7 \times 2^3 &\quad \times \\
3^2 \times (2 * 5) \times 11 \times (3 * 2^2) \times 13 \times (2 * 7) \times (3 * 5) &\quad \times \\
(2^4) \times 17 \times (2 * 3^2) \times 19 \times (2^2 * 5) \times (3 * 7) \times (2 * 11) &\quad = \\
2^{19} \times 3^9 \times 5^4 \times 7^3 \times 11^2 \times 13 \times 17 \times 19. &
\end{aligned}$$

1.07. *Show that, if* 2^a *is the highest power of* 2 *which divides* $n!$ *then* a *lies between* $n-1$ *(higher bound) and* $n - \lfloor \log_2(n+1) \rfloor$ *, (lower bound) where* $\lfloor \log_2(n+1) \rfloor$ *is the exponent of the greatest power of* 2 *not greater than* $n+1$*.*

Discussion: Let us take a look at some numbers. The first column contains some integers from $n = 4 = 2^2$ to $n = 16 = 2^4$. The second column shows the factorials, $n!$, one can see how fast they grow.

n	$n!$	$\lfloor \log_2(n+1) \rfloor$	*lower*	a	*upper*	2^a
4	24	2	2	3	3	2^3
5	120	2	3	3	4	2^3
6	720	2	4	4	5	2^4
7	5,040	3	4	4	6	2^4
8	40,320	3	5	7	7	2^7
9	362,880	3	6	7	8	2^7
10	3,628,800	3	7	8	9	2^8
11	39,916,800	3	8	8	10	2^8
12	479,001,600	3	9	10	11	2^{10}
13	6,227,020,800	3	9	10	11	2^{10}
14	87,178,291,200	3	9	10	11	2^{10}
15	1,307,674,368,000	3	9	10	11	2^{10}
16	20,922,789,888,000	4	12	15	15	2^{15}

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2		*		*		*		*		*		*		*		*
4				*				*				*				*
8								*								*
16																*

1.11 If $2^n - 1$ is prime, show that n is prime. Is the converse true? (Davenport-Guy)

Prime numbers of the form $2^n - 1$ are the Mersenne primes. If $2^n - 1$ is prime then n is prime, for suppose to the contrary that $n = kl$ compound number where k is a proper divisor of n then

$$(2^n - 1) = (2^k - 1)(2^{(l-1)k} + 2^{(l-2)k} + 2^{(l-3)k} \ldots + 1),$$

so $2^n - 1$ has a proper divisor. Therefore the primality of $2^n - 1$ is reduced to the primality of $2^p - 1$, where p is prime. Standard notation for Mersenne prime is

$$M_p = 2^p - 1.$$

Hua gives a list of Mersenne primes. Examples:

$$p = 2, 3, 5, 7, 13$$

$M_2 = 2(2^2 - 1) = 6; M_3 = 2^2(2^3 - 1) = 28; M_5 = 2^4(2^5 - 1) = 496$

Not all odd primes are Mersenne primes. Observe that

$(2^{11} - 1) = 2047 = 23 * 89.$

Calculation by Fermat's method:

$45^2 < 2047 < 46^2$

$$\begin{aligned}
46^2 - 2047 &= 69 \\
47^2 - 2047 &= 162 \\
48^2 - 2047 &= 257 \\
49^2 - 2047 &= 354 \\
50^2 - 2047 &= 453 \\
51^2 - 2047 &= 554 \\
52^2 - 2047 &= 657 \\
53^2 - 2047 &= 762 \\
54^2 - 2047 &= 869 \\
55^2 - 2047 &= 978 \\
56^2 - 2047 &= 1089 = 33^2 \\
56^2 - 33^2 &= 2047 \\
(56 + 33) * (56 - 33) &= 2047 \\
89 * 23 &= 2047
\end{aligned}$$

1.12 *If $2^n + 1$ is prime, show that n is a power of 2. Is the converse true? (Davenport-Guy)*

Suppose - if possible - that $2^n + 1$ is prime, and $n = q * r$, where q is an odd divisor of n.

Then

$2^{qr} + 1 = (2^r + 1)(1 - 2^r + 2^{2r} - \ldots + 2^{(q-1)r}).$

Verification

$(2^r * 1 - 2^r * 2^r + 2^r * 2^{2r} - \ldots + 2^r * 2^{(q-1)r}) + (1 - 2^r + 2^{2r} - \ldots + 2^{(q-1)r}) =$

$(2^r - 2^{2r} + 2^{2r+1} - \ldots - 2^{(q-1)r} + 2^{qr}) + (1 - 2^r + 2^{2r} - \ldots + 2^{(q-1)r}) =$

$1 + (2^r - 2^r) - (2^{2r} - 2^{2r}) + \ldots - (2^{(q-1)r} - 2^{(q-1)r}) + 2^{qr} =$

$1 + 2^{qr}.$

1.13 *If P_1, P_2, are even perfect numbers with $6 < P_1 < P_2$ show that $P_2 > 16P_1$. (Davenport-Guy)*

Let n be a positive integer and let $\sigma(n)$ denote the sum of the divisors of n. By the fundamental theorem of Arithmetic any natural number (positive integer) can be represented in one and only one way as a product of primes. Thus $n = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$ with primes $p_1, p_2 \ldots p_s$ and positive integers $a_1, a_2 \ldots a_s$. First, we show

$$\sigma(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_s^{a_s+1} - 1}{p_s - 1}.$$

All divisors of n are of the form

$$p_1^{x_1} p_2^{x_2} \ldots p_s^{x_s}$$

where

$$0 \le x_1 \le a_1,\ 0 \le x_2 \le a_2, \ldots\ 0 \le x_s \le a_s.$$

Moreover, the summation of the finite geometric series is

$$p_1^0 + p_1^1 + p_1^2 \ldots p_1^{a_1} = \frac{p_1^{a_1+1} - 1}{p_1 - 1}.$$

Therefore

$$\begin{aligned}
\sigma(n) &= \sum_{x_1=0}^{x_1=a_1} \sum_{x_2=0}^{x_2=a_2} \cdots \sum_{x_s=0}^{x_s=a_s} p_1^{x_1} p_2^{x_2} \ldots p_s^{x_s} \\
&= \sum_{x_1=0}^{x_1=a_1} p_1^{x_1} \sum_{x_2=0}^{x_2=a_2} p_2^{x_2} \cdots \sum_{x_s=0}^{x_s=a_s} p_s^{x_s} \\
&= \frac{p_1^{a_1+1} - 1}{p_1 - 1} \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_s^{a_s+1} - 1}{p_s - 1}.
\end{aligned}$$

It also follows for numbers m and n, $(m, n) = 1$ that

$$\sigma(mn) = \sigma(m)\sigma(n)$$

and $\sigma(n)$ is a multiplicative (arithmethical) function.

5.4 Tutorial 3.

- Higher Arithmetic
- *Spartan Old School* Γ
- Last revision September 30, 2014

Exercises

6. Examples of *prime factorization* :
i) N=999

$$\begin{aligned} 999:3 &= 333 \\ 333:3 &= 111 \\ 111:3 &= 37 \\ 37 &= 37 \end{aligned}$$

$N = 3 * 3 * 3 * 37.$

ii) N=1001, by *Fermat's method* .

$31^2 < 1001 < 32^2$

Searching for a complete square:

$$\begin{aligned} 32^2 - N &= 23 \\ 33^2 - N &= 88 \\ 34^2 - N &= 155 \\ 35^2 - N &= 224 \\ 36^2 - N &= 295 \\ 37^2 - N &= 368 \\ \ldots &= \ldots \\ 45^2 - N &= 1024 = 32^2 \\ 45^2 - 32^2 &= N.\surd \end{aligned}$$

$$N = (45 + 32)(45 - 32) = 77 * 13 = 7 * 11 * 13.$$

iii) N=1729, by *Fermat's method* .

$$41^2 < 1729 < 42^2$$

$$\begin{aligned} 42^2 - N &= 35 \\ \ldots &= \ldots \\ 55^2 - N &= 1296 = 36^2 \\ 55^2 - 36^2 &= N.\surd \end{aligned}$$

$$N = (55 + 36)(55 - 36) = 91 * 19 = 7 * 13 * 19.$$

iv) N=11111 The smallest prime factor in N (if any) is not greater than $\sqrt{N}$. Thus we consider the primes up to $\sqrt{N} < 106$:

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97
101	103	107	...	...
...	...	...	...	...
...	...	271	...	...

The first prime that divides N is 41.

$$11111 : 41 = 271$$

It turns out that 271 is also a prime, therefore

$$N = 41 * 271.$$

7. Examples of *Euclidean algorithm* : Find the *highest common factor* of

$i)$ $6188, 4709$ $ii)$ $81719, 52003, 33649, 30107.$

Proof: i)

$$\begin{aligned} 6188 &= 1 * 4709 + 1479 \\ 4709 &= 3 * 14709 + 272 \\ 1479 &= 5 * 272 + 119 \\ 272 &= 2 * 119 + 34 \\ 119 &= 3 * 34 + 17 \\ 34 &= 2 * 17 + 0 \end{aligned}$$

The highest common factor of 6188, 4709 is 17, the last positive remainder. Next, express the h.c.f as a linear (Diophantine) combination of $a = 6188$ and $b = 4709$.

$$\begin{aligned} 1479 &= a - b \\ b &= 3 * (a - b) + 272 \\ 272 &= 4b - 3a \\ 119 &= 16a - 21b \\ 34 &= 46b - 35a \\ 17 &= 121a - 159b. \end{aligned}$$

Check:

$$121a - 159b = 121 * 6188 - 159 * 4709 = 748748 - 748731 = 17.\surd$$

ii)

$$m(a_1, a_2) = m_2; m(m_2, a_3) = m_3; m(m_3, a_4) = m_4$$

$$\begin{aligned} 81719 &= 1 * 52003 + 29716 \\ 52003 &= 1 * 29716 + 22287 \\ 29287 &= 1 * 22287 + 7429 \\ 22287 &= 3 * 7429 \end{aligned}$$

$$\begin{aligned} 81719 : 7429 &= 11 \\ 52003 : 7429 &= 7 \\ m_2 &= 7429. \end{aligned}$$

$$\begin{aligned}
33649 &= 4*7429+3933\\
7429 &= 1*3933+3496\\
3933 &= 1*3496+437\\
3496 &= 8*437
\end{aligned}$$

$$\begin{aligned}
81719:437 &= 187\\
52003:437 &= 119\\
33649:437 &= 77\\
m_3 &= 437.
\end{aligned}$$

$$\begin{aligned}
30107 &= 68*437+391\\
437 &= 1*391+46\\
391 &= 8*46+23\\
46 &= 2*23.
\end{aligned}$$

$$m_4 = 23.$$

8. Show that

$$i)\ n(n+1)(2n+1)$$

is divisible by 6.

Proof : Write $N = n(n+1)(2n+1)$. If n is odd then $n+1$ is even and N is divisible by 2. If n is even then N is divisible by 2.

If $n \equiv 0\ (mod 3)$ then N is divisible by 3. If $n \equiv 1\ (mod 3)$ then $2n+1 \equiv 0\ (mod 3)$ and N is divisible by 3. If $n \equiv 2$ then $n+1 \equiv 0\ (mod 3)$ and N is divisible by 3. Since n is in exactly one of the above equivalence classes N is divisible by 3.

Therefore N is divisible by both 2 and 3, thus by 6.

8. Show that

$$ii)\ m + \frac{1}{2}(m+n-1)(m+n-2)$$

runs through the whole set of positive integers without repetition as m and n run through the set of all positive integers.

Discussion: Consider the lattice points (points with integer coordinates) in the positive quadrant of the Cartesian coordinate system. As m and n run through the whole set of positive integers the ordered pair (m, n) runs through the lattice points :

$$A = \{(m, n),\ m = 1, 2, \ldots;\ n = 1, 2, \ldots\}.$$

Write

$$f(m, n) = m + \frac{1}{2}(m + n - 1)(m + n - 2).$$

The *range* of this function is A since (m, n) takes on every possible pair of positive integers. Further, f is an integer because exactly one of the two consecutive numbers $(m + n - 1)$,$(m + n - 2)$ is even. So the *image* of f is a subset of the set of positive integers. We will show that this map is in fact, bijective: it maps the lattice points of the positive quadrant onto the set of positive integers. First, we shall consider a subset of A. Take

$$A_6 = \{(m, n),\ m = 1, 2, \ldots, 5;\ n = 1, 2, \ldots, 5;\ m + n \leq 6\}.$$

The points belonging to A_6 are on the lower left corner of A. A_6 has a triangular shape, there are 15 points in it, and A_6 can be covered by

$$\begin{aligned}
D_1 &= \{(1, 1)\} \\
D_2 &= \{(1, 2), (2, 1)\} \\
D_3 &= \{(1, 3), (2, 2), (3, 1)\} \\
D_4 &= \{(1, 4), (2, 3), (3, 2), (4, 1)\} \\
D_5 &= \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}.
\end{aligned}$$

The sets $D_1, D_2, \ldots, D_5$ provide a *cover* of A_6:

$$D_1 \cup D_2 \cup, \ldots \cup D_5 = A_6;\ D_1 \cap D_2 \cap, \ldots \cap D_5 = \emptyset.$$

Notice also that $D_1, D_2, \ldots, D_5$ look like "slanting diagonals" on latice points. Another interesting feature is that

$$\frac{1}{2}(m + n - 1)(m + n - 2) = const$$

on each D_i. Let us calculate $f(m,n)$ on A_6 :

$$f(1,1) = 1 + \frac{1}{2}(1+1-1)(1+1-2) = 1,$$

$$f(1,2) = 1 + \frac{1}{2}(1+2-1)(1+2-2) = 2,$$

$$f(2,1) = 2 + \frac{1}{2}(2+1-1)(2+1-2) = 3,$$

$$D_3: \ f(1,3) = 4, \ f(2,2) = 5, \ f(3,1) = 6,$$

$$D_4: \ f(1,4) = 7, \ f(2,3) = 8, \ f(3,2) = 9, f(4,1) = 10,$$

the pattern is discernible,

$$D_5: \ f(1,5) = 11, \ f(2,4) = 12, \ f(3,3) = 13; f(4,2) = 14, \ f(5,1) = 15.$$

Define

$$\begin{aligned} C_1 &= \{1\} \\ C_2 &= \{2,3\} \\ C_3 &= \{4,5,6\} \\ C_4 &= \{7,8,9,10\} \\ C_5 &= \{11,12,13,14,15\}. \end{aligned}$$

Note that the sets $C_1, C_2, \ldots, C_5$ provide a *cover* of $\{1,2,3,\ldots,15\}$, moreover, each C_i is an ordered set of positive integers; the minimal element of each C_k is the sum of the first $k-1$ positive integers plus 1, the maximal element of each C_k is the sum of the first k positive integers, $i > 1$. So having partitioned A and $\{1,2,3,\ldots,15\}$ we note that f maps each D_i onto C_i, $i = 1,2\ldots5$. Therefore f is a *bijection* , and f^{-1} induces a linear order on A_6.

Proof: Extend the above construction by adjoining successive "slanting diagonals". Then each positive number will be associated with exactly one ordered pair of positive integers, (m,n).

5.5 Tutorial 4.

- Higher Arithmetic
- *Spartan Old School* Γ
- Last revision September 30, 2014

1.14 If p, q are odd primes, show that $p^a q^b$ can not be perfect.
Proof: The number n is *perfect* if the sum of its divisors, including 1, but excluding n, is equal to n itself.

$$\begin{aligned} n &= p^a q^b \\ \sigma(n) &= 2n \\ \sigma(n) &= \left(1 + p + p^2 + \ldots + p^a\right)\left(1 + q + q^2 + \ldots + q^b\right) \end{aligned}$$

Suppose - if possible - that

$$\begin{aligned} \sigma(p^a q^b) &= 2p^a q^b \\ \frac{p^{a+1} - 1}{p - 1}\frac{q^{b+1} - 1}{q - 1} &= 2p^a q^b \\ \frac{p^{a+1} - 1}{p^a}\frac{q^{b+1} - 1}{q^b} &= 2(p-1)(q-1) \\ \frac{\left(p - \frac{1}{p^a}\right)}{(p-1)}\frac{\left(q - \frac{1}{q^b}\right)}{(q-1)} &= 2 \\ \frac{p}{(p-1)}\frac{q}{(q-1)} &< 2 \end{aligned}$$

Look at the first two primes $p = 3, q = 5$

$$\frac{3}{2} \times \frac{5}{4} = \frac{15}{8} < 2.$$

Notice further that all odd primes are a subset of the odd numbers $2n-1 > 1$, $n = 1, 2, \ldots$ and the sequence

$$\frac{3}{2} > \frac{5}{4} > \ldots > \frac{2n-1}{2n}$$

is strictly monotone decreasing. Let r, s be two odd primes $r < s$. Then

$$\frac{r}{(r-1)}\frac{s}{(s-1)} \leq \frac{3}{2} \times \frac{5}{4} = \frac{15}{8} < 2.$$

(Elementary, my dear Watson .)

Remarks:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^2 + 3^2 + 5^2 \ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

For proofs by induction see *Tutorials on Higher Arithmetic*. A positive integer n is called a *perfect number* if $\sigma(n) = 2n$. Examples:

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7$$

$$\begin{array}{r|l} 496 & 2 \\ 248 & 2 \\ 124 & 2 \\ 62 & 2 \\ 31 & 31 \\ 1 & \end{array}$$

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248.$$

Let $p = 2^n - 1$ be a prime. Then

$$\frac{1}{2}p(p+1) = 2^{n-1}(2^n - 1)$$

is perfect. It is an even perfect number. Note that $p + 1 = 2^n$, hence $\frac{1}{2}(p+1) = 2^{n-1}$ and the sum of divisors of the latter number is

$$\sigma(\frac{1}{2}(p+1)) = 1 + 2 + \ldots + 2^{n-1} = 2^n - 1.$$

This shows that a power of 2 is not a perfect number. Furthermore,

$$\sigma(p) = p + 1,$$

since $\sigma(n)$ is a multiplicative function and $(p, 2^{n-1}) = 1$,

$$\sigma(\frac{1}{2}p(p+1)) = \sigma(p)\sigma(\frac{1}{2}(p+1)).$$

$$\begin{aligned}\sigma(\frac{1}{2}p(p+1)) &= \frac{2^n-1}{2-1}\frac{p^2-1}{p-1} \\ &= (2^n-1)(p+1) \\ &= p(p+1).\end{aligned}$$

This shows that $\frac{1}{2}p(p+1)$ is perfect.

Moreover, every even perfect numbers is of this form. To see this let a be an even perfect number

$$a = 2^{n-1}u,\ u > 1, (2, u) = 1.$$

We have shown above that a a is not a power of 2. Therefore u is equal to a prime or a product of primes all greater than 2. By construction

$$2^n u = 2a,$$

and a is a perfect number

$$2a = \sigma(a).$$

$$\sigma(a) = \sigma(2^{n-1}u) = \sigma(2^{n-1})\sigma(u) = \frac{2^n-1}{2-1}\sigma(u)$$

Connecting the equations gives

$$2^n u = \frac{2^n-1}{2-1}\sigma(u).$$

$$\sigma(u) = \frac{2^n u}{2^n-1}$$

$$\frac{2^n u}{2^n-1} = \frac{2^n u + u - u}{2^n-1} = \frac{(2^n u - u) + u}{2^n-1} = u + \frac{u}{2^n-1}$$

Therefore

$$\sigma(u) = u + \frac{u}{2^n-1}.$$

The left hand side is an integer, u is an integer so the second term on the right hand is an integer. Furthermore

$$\frac{u}{\frac{u}{2^n - 1}} = 2^n - 1.$$

Thus $\sigma(u)$ is the sum of two divisors, one is u the other is $\frac{u}{2^n - 1}$. Suppose - if possible - that u has proper divisors $\omega_1, \omega_2 \ldots \omega_s$, and all divisors of u are listed in strictly monotone descending order:

$$\sigma(u) = u + \omega_1 + \omega_2 + \ldots + \omega_s + 1$$

If

$$\frac{u}{2^n - 1} = u$$

then

$$\sigma(u) = u + u = u + 1,$$

$u = 1$ which contradicts the assumption $u > 1$. If for some fixed i, $1 \leq i \leq s$

$$\frac{u}{2^n - 1} = \omega_i,$$

then

$$\sigma(u) = u + \frac{u}{2^n - 1} + 1.$$

But this is a contradiction. Therefore there are no $\omega_1, \omega_2 \ldots \omega_s$. Hence

$$\frac{u}{2^n - 1} = 1$$

$$\sigma(u) = u + \frac{u}{2^n - 1} = u + 1,$$

u has two divisors, and u is an odd prime.

$$u = 2^n - 1.$$

Let us collect our results. Every even perfect number has the form of

$$2^{p-1}(2^p - 1)$$

where p is a prime and $M_p = 2^p - 1$ is a Mersenne prime (see Exercise 1.11.) If M_p and M_q are two consecutive Mersenne primes $2 < p \leq p+2 \leq q$ then

$$\frac{2^{q-1}(2^q - 1)}{2^{p-1}(2^p - 1)} \leq \frac{2^{p+1}(2^{p+2} - 1)}{2^{p-1}(2^p - 1)} \leq \frac{4 * 2^{p-1} 4 * (2^p - 2^{-2})}{2^{p-1}(2^p - 1)} \leq 16 \left(\frac{2^p - 2^{-2}}{2^p - 1} \right) < 16$$

because

$$\left(\frac{2^p - 2^{-2}}{2^p - 1} \right) < 1.$$

The End

5.6 Assignment 1.

- Combinatorial Analysis
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision September 30, 2014

Problems

I 23. The total number N of non-negative integral solutions of the following Diophantine equations

$$x + 2y = n - 1;\ 2x + 3y = n - 3;\ 3x + 4y = n - 5;\ \ldots$$

is smaller than $n+2$; moreover the difference $n+2-N$ is equal to the number of divisors of $n+2$.

Remark: The total number of divisors of number n (including 1 and n) is denoted by $d(n)$. So if $n = p^a q^b \ldots$ with $p, q \ldots$ distinct primes then

$$d(n) = (a+1)(b+1)\ldots$$

Discussion: The left-hand sides of the Diophantine equations are the same as in **I. 22.** Skipping the discussion on generators and formal summations we have

Claim I: the total number N of non-negative integral solutions is the coefficient of ξ^{n-1} in the expansion of

$$S(\xi) = \sum_{\nu=0}^{\nu=\infty} \frac{\xi^{2\nu}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}.$$

Proof of Claim I: The number of solutions to the first Diophantine equation is equal to A_{n-1} in the expansion

$$\frac{1}{(1-\xi^1)(1-\xi^2)} = \sum_{\nu=0}^{\nu=\infty} A_\nu \xi^\nu.$$

Similarly, the numbers of solutions to the second and third Diophantine equations are given by B_{n-3} in the expansion

$$\frac{1}{(1-\xi^2)(1-\xi^3)} = \sum_{\nu=0}^{\nu=\infty} B_\nu \xi^\nu$$

and C_{n-5}

$$\frac{1}{(1-\xi^3)(1-\xi^4)} = \sum_{\nu=0}^{\nu=\infty} C_\nu \xi^\nu,$$

respectively. Therefore the combined number of the non-negative integral solutions to the first three equations

$$x + 2y = n - 1;\ 2x + 3y = n - 3; 3x + 4y = n - 5;$$

is $A_{n-1} + B_{n-3} + C_{n-5}$.

Multiplication by ξ^2 and ξ^4 shifts B_ν to $B_{\nu+2}$ and C_ν to $C_{\nu+4}$, respectively, and the required number is the coefficient of ξ^{n-1} in the final expansion, whenever $n \geq 5$

Write

$$\frac{1}{(1-\xi^1)(1-\xi^2)} + \frac{\xi^2}{(1-\xi^2)(1-\xi^3)} + \frac{\xi^4}{(1-\xi^3)(1-\xi^4)} =$$

$$\sum_{\nu=0}^{\nu=\infty} A_\nu \xi^\nu + \xi^2 \sum_{\nu=0}^{\nu=\infty} B_\nu \xi^\nu + \xi^4 \sum_{\nu=0}^{\nu=\infty} C_\nu \xi^\nu =$$

$$\sum_{\nu=0}^{\nu=\infty} A_\nu \xi^\nu + \sum_{\nu=0}^{\nu=\infty} B_\nu \xi^{\nu+2} + \sum_{\nu=0}^{\nu=\infty} C_\nu \xi^{\nu+4} = \sum_{\nu=0}^{\nu=\infty} (A_\nu + B_{\nu-2} + C_{\nu-4}) \xi^\nu,$$

$$B_{-2} = B_{-1} = C_{-4} = C_{-3} = C_{-2} = C_{-1} = 0.$$

Therefore, in this manner, we can build a solution that includes the non-negative integral solutions to all Diophantine equations :

$$S(\xi) = \frac{1}{(1-\xi^1)(1-\xi^2)} + \frac{\xi^2}{(1-\xi^2)(1-\xi^3)} + \frac{\xi^4}{(1-\xi^3)(1-\xi^4)} + \ldots$$

Claim II:

$$S(\xi) = \sum_{\nu=0}^{\nu=\infty} \frac{\xi^{2\nu}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})} = \frac{1}{1-\xi} \sum_{\nu=0}^{\nu=\infty} \xi^{\nu-1} \left(\frac{1}{1-\xi^{\nu+1}} - \frac{1}{1-\xi^{\nu+2}} \right).$$

Check:

$$\frac{\xi^{\nu-1}}{1-\xi}\left(\frac{1}{1-\xi^{\nu+1}}-\frac{1}{1-\xi^{\nu+2}}\right)=$$

$$\frac{\xi^{\nu-1}}{1-\xi}\left(\frac{1-\xi^{\nu+2}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}-\frac{1-\xi^{\nu+1}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}\right)=$$

$$\frac{\xi^{\nu-1}}{1-\xi}\left(\frac{\xi^{\nu+1}-\xi^{\nu+2}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}\right)=\frac{\xi^{\nu-1}}{1-\xi}\left(\frac{\xi^{\nu+1}(1-\xi)}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}\right)=$$

$$\frac{\xi^{2\nu}}{(1-\xi^{\nu+1})(1-\xi^{\nu+2})}.\surd$$

Claim III:

$$\begin{aligned}
S(\xi) &= \frac{1}{1-\xi}\sum_{\nu=0}^{\nu=\infty}\xi^{\nu-1}\left(\frac{1}{1-\xi^{\nu+1}}-\frac{1}{1-\xi^{\nu+2}}\right)\\
&= \frac{1}{1-\xi}\left[\xi^{-1}\left(\frac{1}{1-\xi^{1}}-\frac{1}{1-\xi^{2}}\right)\right]_{(\nu=0)}\\
&+ \frac{1}{1-\xi}\left[\xi^{0}\left(\frac{1}{1-\xi^{2}}-\frac{1}{1-\xi^{3}}\right)\right]_{(\nu=1)}\\
&+ \frac{1}{1-\xi}\left[\xi^{1}\left(\frac{1}{1-\xi^{3}}-\frac{1}{1-\xi^{4}}\right)\right]_{(\nu=2)}\\
&+ \frac{1}{1-\xi}\left[\xi^{2}\left(\frac{1}{1-\xi^{4}}-\frac{1}{1-\xi^{5}}\right)\right]_{(\nu=3)}+\dots\\
&+ \frac{1}{1-\xi}\left[\xi^{n-1}\left(\frac{1}{1-\xi^{n+1}}-\frac{1}{1-\xi^{n+2}}\right)\right]_{(\nu=n)}\\
&+ \frac{1}{1-\xi}\left[\xi^{n}\left(\frac{1}{1-\xi^{n+2}}-\frac{1}{1-\xi^{n+3}}\right)\right]_{(\nu=n+1)}+\dots
\end{aligned}$$

Next combine the terms of

$$\frac{1}{1-\xi^{\nu+2}}$$

from two consecutive terms $\nu, \nu+1$, here found along slanting diagonals:

$$\begin{aligned} S(\xi) &= \frac{1}{\xi}\frac{1}{(1-\xi)^2} \\ &+ \frac{1}{1-\xi}\left(\xi^0\frac{1}{1-\xi^2} - \xi^{-1}\frac{1}{1-\xi^2}\right)_{(\nu=0,1)} \\ &+ \frac{1}{1-\xi}\left(\xi^1\frac{1}{1-\xi^3} - \xi^{0}\frac{1}{1-\xi^3}\right)_{(\nu=1,2)} \\ &+ \frac{1}{1-\xi}\left(\xi^2\frac{1}{1-\xi^4} - \xi^{1}\frac{1}{1-\xi^4}\right)_{(\nu=2,3)} + \ldots \\ &+ \frac{1}{1-\xi}\left(\xi^n\frac{1}{1-\xi^{n+2}} - \xi^{n-1}\frac{1}{1-\xi^{n+2}}\right)_{(\nu=n,n+1)} + \ldots \end{aligned}$$

$$\begin{aligned} S(\xi) &= \frac{1}{\xi}\frac{1}{(1-\xi)^2} \\ &+ \frac{(\xi^0-\xi^{-1})}{1-\xi}\left(\frac{1}{1-\xi^2}\right) + \frac{(\xi^1-\xi^{0})}{1-\xi}\left(\frac{1}{1-\xi^3}\right) \\ &+ \frac{(\xi^2-\xi^{1})}{1-\xi}\left(\frac{1}{1-\xi^4}\right) + \ldots + \frac{(\xi^n-\xi^{n-1})}{1-\xi}\left(\frac{1}{1-\xi^{n+2}}\right)\ldots \end{aligned}$$

Noting that

$$\left(\frac{1}{\xi^3}\frac{\xi}{1-\xi} - \frac{1}{\xi^3}\frac{\xi}{1-\xi}\right) = 0$$

and

$$\left(\frac{\xi^3}{\xi^3}\right) = 1$$

we have

$$\begin{aligned} S(\xi) &= \frac{1}{\xi}\frac{1}{(1-\xi)^2} + \left(\frac{1}{\xi^3}\frac{\xi}{1-\xi} - \frac{1}{\xi^3}\frac{\xi}{1-\xi}\right) \\ &+ \left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^0-\xi^{-1})}{1-\xi}\left(\frac{1}{1-\xi^2}\right) + \left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^1-\xi^{0})}{1-\xi}\left(\frac{1}{1-\xi^3}\right) \\ &+ \left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^2-\xi^{1})}{1-\xi}\left(\frac{1}{1-\xi^4}\right) + \ldots + \left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^n-\xi^{n-1})}{1-\xi}\left(\frac{1}{1-\xi^{n+2}}\right)\ldots \end{aligned}$$

By examining $S(\xi)$ term by term we obtain
First:

$$\begin{aligned}\frac{1}{\xi}\frac{1}{(1-\xi)^2}+\frac{1}{\xi^3}\left(\frac{\xi}{1-\xi}\right) &= \frac{1}{\xi^2}\frac{\xi}{(1-\xi)^2}+\frac{1}{\xi^2}\left(\frac{1}{1-\xi}\right)= \\ \frac{1}{\xi^2}\left(\frac{\xi}{(1-\xi)^2}+\left(\frac{1}{1-\xi}\right)\right) &= \frac{1}{\xi^2}\left(\frac{\xi}{(1-\xi)^2}+\frac{1-\xi}{(1-\xi)^2}\right)=\frac{1}{\xi^2}\frac{1}{(1-\xi)^2}.\end{aligned}$$

Second:

$$-\frac{1}{\xi^3}\frac{\xi}{1-\xi}$$

Third:

$$\begin{aligned}\left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^0-\xi^{-1})}{1-\xi}\left(\frac{1}{1-\xi^2}\right) &= \left(\frac{\xi}{\xi^3}\right)\frac{(\xi^0-\xi^{-1})}{1-\xi}\left(\frac{\xi^2}{1-\xi^2}\right)= \\ \left(\frac{1}{\xi^3}\right)\frac{(\xi-1)}{1-\xi}\left(\frac{1}{1-\xi^2}\right) &= -\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^2}{1-\xi^2}\right).\end{aligned}$$

Fourth:

$$\left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^1-\xi^0)}{1-\xi}\left(\frac{1}{1-\xi^3}\right) = -\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^3}{1-\xi^3}\right)\ldots$$

N-th:

$$\begin{aligned}\left(\frac{\xi^3}{\xi^3}\right)\frac{(\xi^{n-3}-\xi^{n-4})}{1-\xi}\left(\frac{1}{1-\xi^{n-1}}\right) &= \left(\frac{1}{\xi^3}\right)\frac{\xi-1}{1-\xi}\left(\frac{\xi^{n-1}}{1-\xi^{n-1}}\right)= \\ &-\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^{n-1}}{1-\xi^{n-1}}\right).\end{aligned}$$

$$\begin{aligned}S(\xi) &= \frac{1}{\xi^2}\frac{1}{(1-\xi)^2} \\ &- \left(\frac{1}{\xi^3}\right)\left(\frac{\xi}{1-\xi}\right)-\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^2}{1-\xi^2}\right) \\ &- \left(\frac{1}{\xi^3}\right)\left(\frac{\xi^3}{1-\xi^3}\right)\ldots-\left(\frac{1}{\xi^3}\right)\left(\frac{\xi^{n-1}}{1-\xi^{n-1}}\right)\ldots \\ &= \frac{1}{\xi^2}\frac{1}{(1-\xi)^2}-\left(\frac{1}{\xi^3}\right)\sum_{\nu=1}^{\nu=\infty}\frac{\xi^\nu}{1-\xi^\nu}.\end{aligned}$$

Next, we examine the main term

$$\begin{aligned}
\frac{1}{\xi^2}\frac{1}{(1-\xi)^2} &= \frac{1}{\xi^2} * \frac{1}{(1-\xi)} * \frac{1}{(1-\xi)} \\
&= \frac{1}{\xi^2} * (1+\xi+\xi^2+\ldots) * (1+\xi+\xi^2+\ldots) \\
&= \frac{1}{\xi^2} * (1+2\xi+3\xi^2+\ldots+(n+1)\xi^n+(n+2)\xi^{n+1}+(n+3)\xi^{n+2}) \\
&= (\ldots+(n+1)\xi^{n-2}+(n+2)\xi^{n-1}+(n+3)\xi^n+\ldots)
\end{aligned}$$

This shows that the coefficient of ξ^{n-1} is $(n+2)$ in the main term. Now we come to the secondary term without $\left(\frac{1}{\xi^3}\right)$.

Claim IV:

$$\sum_{\nu=1}^{\nu=\infty} \frac{\xi^\nu}{1-\xi^\nu} = \tau(1)\xi+\tau(2)\xi^2+\ldots+\tau(\nu)\xi^\nu+$$

where $\tau(\nu)$ denotes the number of divisors of ν.

Discussion of Claim IV: Let us pick a number - say 6 - and let us see what contributes to the coefficient of ξ^6.

$$\begin{aligned}
\frac{\xi^1}{1-\xi^1} &= \xi+\xi^2+\xi^3+\xi^4+\xi^5+\underline{\xi^6}+\xi^7+\ldots \\
\frac{\xi^2}{1-\xi^2} &= \xi^2+\xi^4+\underline{\xi^6}+\xi^8+\ldots \\
\frac{\xi^3}{1-\xi^3} &= \xi^3+\underline{\xi^6}+\xi^9+\ldots \\
\frac{\xi^6}{1-\xi^6} &= \underline{\xi^6}+\xi^{12}\ldots
\end{aligned}$$

Clearly, expansions for $\nu = 4, 5$, or $\nu \geq 7$ would not contribute. Therefore the coefficient of ξ^6 is equal to 4, the number of ν -s that divide 6. Further, pick another number - say 7 - . Only two expansions have ξ^7 in them:

$$\frac{\xi^1}{1-\xi^1} = \xi+\xi^2+\xi^3+\xi^4+\xi^5+\xi^6+\underline{\xi^7}+\xi^8+\ldots$$

$$\frac{\xi^7}{1-\xi^7} = \underline{\xi^7} + \xi^{14} + \ldots$$

Of course, 7 is a prime, and $\tau(7) = 2$. This not a proof, but we are convinced.

5.7 Assignment 2.

- Classics in Applied Math
- *Ciarlet et al.: Exercises d'analyse numérique matricielle, Morton-Mayers: Num Sol of PDE's*
- Last revision September 30, 2014

Problems

1.1.-1. *Let A be an invertible matrix whose elements, as well as those of A^{-1} are non-negative. Show that there exists a permutation matrix P and a matrix $D = diag(d_i)$, with d_i positive , such that $A = PD$ (converse is obvious).*

Discussion I.: Let us consider a 4×4 matrix A. By construction , A is invertible, thus $|A| \neq 0$. The determinant of A is a sum of products, each product comprises a diagonal; which has four elements, in this example, one from each row and each column. If every product is zero then the determinant is zero and A is not invertible. That is absurd. Therefore at least one diagonal is not zero. For demonstration, let us suppose that

$$a_{13}a_{24}a_{32}a_{41} \neq 0;$$

where

$$a_{13} > 0;\; a_{24} > 0;\; a_{32} > 0;\; a_{41} > 0.$$

Then

$$A = \begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{bmatrix}$$

is an invertible matrix with non-negative elements

$$|A| = a_{13}a_{24}a_{32}a_{41} > 0.$$

Next, we find B, the inverse pair of A by costructing the adjoint first. Recall that the adjoint of A is the transposed matrix of cofactors of A and

$$B = \frac{1}{detA} adjA$$

is the inverse of A. The non-zero elements of B are calculated as follows:

$$b_{14} = \frac{a_{13}a_{24}a_{32}}{a_{13}a_{24}a_{32}a_{41}} = \frac{1}{a_{41}}$$

$$b_{23} = \frac{a_{13}a_{24}a_{41}}{a_{13}a_{24}a_{32}a_{41}} = \frac{1}{a_{32}}$$

$$b_{31} = \frac{a_{24}a_{32}a_{41}}{a_{13}a_{24}a_{32}a_{41}} = \frac{1}{a_{13}}$$

$$b_{42} = \frac{a_{13}a_{32}a_{41}}{a_{13}a_{24}a_{32}a_{41}} = \frac{1}{a_{24}}$$

$$\begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{13}b_{31} & 0 & 0 & 0 \\ 0 & a_{24}b_{42} & 0 & 0 \\ 0 & 0 & a_{32}b_{23} & 0 \\ 0 & 0 & 0 & a_{41}b_{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore if matrix A has one and only one diagonal with positive elements and all other elements are zero then A is invertible and its inverse B has one and only one diagonal with positive elements and all other elements of B are zero.

Discussion II.: So far we have obtained a pair of matrices A, B non-negative, $AB = I$. Write

$$\begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{41} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{42} & 0 & 0 & 0 \\ 0 & b_{31} & 0 & 0 \\ 0 & 0 & b_{23} & 0 \\ 0 & 0 & 0 & b_{14} \end{bmatrix}.$$

Both A and B can be written as a product of a permutation matrix and a diagonal matrix. This demonstrates the "converse" or the "obvious" part of the theorem.

We can formulate a tentative theorem: Let $D = diag(d_i)$ be an $n \times n$ matrix with $\forall d_i > 0$ and P be a permutation matrix of the same order. Write

$$D^* = diag(\frac{1}{d_{n-i+1}}),\ A = PD,\ B = P^T D^*.$$

Then A, B are non-negative and

$$AB = I.$$

Discussion III.: Suppose now that matrix A' has a diagonal - say the same as A - plus one positive element y

$$A' = \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & x & 0 & 0 \\ x & 0 & y & 0 \end{bmatrix}$$

Let B' be the inverse of A', assuming that all elements of B' are non-negative.

$$B' = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$A'B' = \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & x & 0 & 0 \\ x & 0 & y & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} =$$

$$\begin{bmatrix} xb_{31} & xb_{32} & xb_{33} & xb_{34} \\ xb_{41} & xb_{42} & xb_{43} & xb_{44} \\ xb_{21} & xb_{22} & xb_{23} & xb_{24} \\ xb_{11} + yb_{31} & xb_{12} + yb_{32} & xb_{13} + yb_{33} & xb_{14} + yb_{34} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There is no cancellation because x, y are positive and b_{ij} are all non-negative, hence

$$xb_{32} = xb_{33} = xb_{34} = 0$$

$$xb_{41} = xb_{43} = xb_{44} = 0$$

$$xb_{21} = xb_{22} = xb_{24} = 0$$

$$xb_{11} + yb_{31} = xb_{12} + yb_{32} = xb_{13} + yb_{33} = 0$$

But then

$$b_{11} = b_{21} = b_{31} = b_{41} = 0$$

one column consists of all zeros, B' is singular, or non-invertible. But this is absurd. Therefore there is no non-zero element y,and matrix A' has a single diagonal. This, of course is not a proof, it is only a demonstration of the main idea of the proof.

2 On PDE's: *The function $u^0(x)$ is defined on $[0, 1]$ by*

$$u^0(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases} \tag{5.1}$$

Show that

$$u^0(x) = \sum_{m=1}^{\infty} a_m \sin(m\pi x)$$

where

$$a_m = (8/m^2\pi^2)\sin(\frac{1}{2}m\pi).$$

(Morton - Mayers)

Proof: a_m is the coefficient in the Fourrier-series expansion of $u^0(x)$

$$a_m = \int_0^1 u^0(x)\sin(m\pi x)dx.$$

Solution by integration on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 0]$

$$a_m = \int_0^{\frac{1}{2}} u^0(x)\sin(m\pi x)dx + \int_{\frac{1}{2}}^1 u^0(x)\sin(m\pi x)dx$$

using the following lemma:

$$\int z\sin(az)dz = \frac{\sin(az)}{a^2} - \frac{z\cos(az)}{a} + C.$$

First integral:

$$\begin{aligned} 2\int_0^{\frac{1}{2}} 2x\sin(m\pi x)dx &= 4\int_0^{\frac{1}{2}} x\sin(m\pi x)dx \\ &= 4\left[\frac{\sin(m\pi x)}{m^2\pi^2} - \frac{x\cos(m\pi x)}{m\pi}\right]_0^{\frac{1}{2}} \\ &= 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi}. \end{aligned}$$

Second integral:

$$\begin{aligned} 2\int_{\frac{1}{2}}^1 2\sin(m\pi x)dx &= 4\left[\frac{-1}{m\pi}\cos(m\pi x)\right]_{\frac{1}{2}}^1 \\ &= \frac{-4}{m\pi}\cos(m\pi 1) + \frac{4}{m\pi}\cos(m\pi\frac{1}{2}) \end{aligned}$$

Third integral:

$$\begin{aligned} 2\int_{\frac{1}{2}}^1 -2x\sin(m\pi x)dx &= -4\int_{\frac{1}{2}}^1 x\sin(m\pi x)dx \\ &= -4\left[\frac{\sin(m\pi x)}{m^2\pi^2} - \frac{x\cos(m\pi x)}{m\pi}\right]_{\frac{1}{2}}^1 \\ &= -4\frac{\sin(m\pi 1)}{m^2\pi^2} + \frac{4\cos(m\pi 1)}{m\pi} \\ &+ 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi}. \end{aligned}$$

From the first and third integral

$$4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} + 4\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2} = 8\frac{\sin(m\pi\frac{1}{2})}{m^2\pi^2}.$$

From the first, second and third

$$-2\frac{\cos(m\pi\frac{1}{2})}{m\pi} + \frac{4}{m\pi}\cos(m\pi\frac{1}{2}) - 2\frac{\cos(m\pi\frac{1}{2})}{m\pi} = 0$$

From the second and third

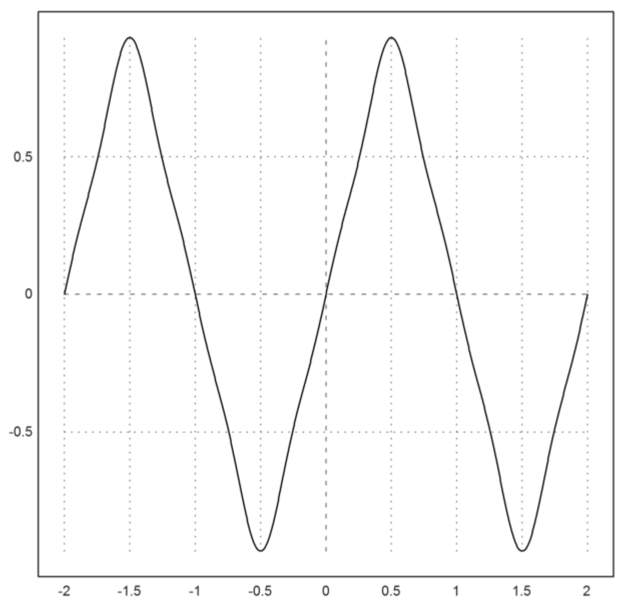
$$-\frac{4}{m\pi}\cos(m\pi 1) + \frac{4}{m\pi}\cos(m\pi 1) = 0$$

and finally

$$4\frac{\sin(m\pi)}{m^2\pi^2} = 0.$$

$$u^0(x) = \frac{8}{\pi^2}\left(\frac{\sin(\pi x)}{1^2} - \frac{\sin(3\pi x)}{3^2} + \frac{\sin(5\pi x)}{5^2}\dots\right)$$

Here is a rough approximation of $u^0(x)$, Fourier sine series truncated after 3 terms:



3. On first-order ODE's: *Let*

$$\frac{dy}{dx} = f(x, y).$$

Display direction field and certain solution curves generated by

$$f(x, y) = \frac{\sin(x)}{\sin(y)}$$

The equation is separable,

$$\frac{dy}{dx} = \frac{\sin(x)}{\sin(y)}$$

$$\sin(y)dy = \sin(x)dx$$

quadrature gives

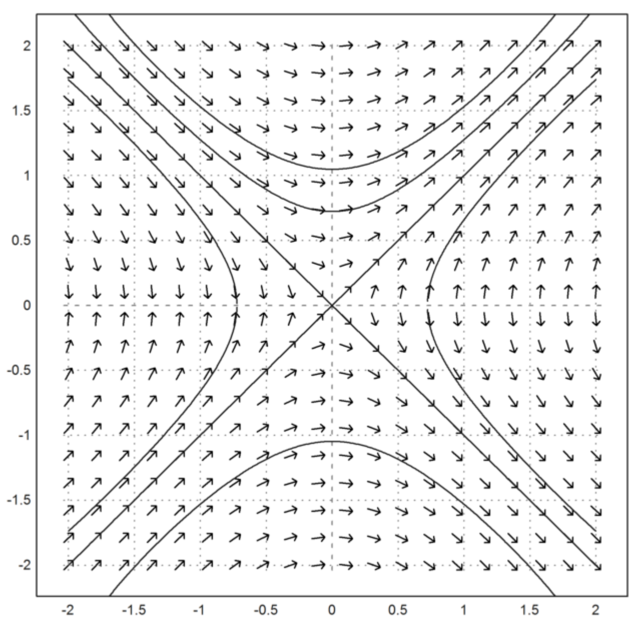
$$-\cos(y) = -cos(x) + C$$

$$y = \arccos(\cos(x) + C).$$

Also

$$y = \pm x$$

with the exception of $(0, 0)$. No solution curve can cut the $y = 0$ axis.



5.8 Miscellaneous Notes

5.8.1 Current interests

Spartan Old School Tutorials: 3 levels, undergraduate standards

Computer Skills: Basic programming with Fortran and C; Math Tools, Graphics, Numerical and Symbolic Computations; Latex typesetting, On-line Tutorials.

Classics in Pure Math: *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis, Konvexer Körper , Vinogradov-Turán*

Classics in Applied Math *Ciarlet, Birkhoff-Rota, Geÿza Freud, Morton-Mayers*

5.8.2 Envoy

"A bicycle, certainly, but not *the* bicycle", he said. " I am familiar with forty-two different impressions left by tyres. This, as you percieve, is a Dunlop, with a patch upon the outer cover. Heidegger's tyres were Palmer's, leaving longitudinal stripes. Aveling, the mathematical master was sure upon the point. Therefore, it is not Heidegger's track."