Pinter Consulting Quarterly Reports New Series No. 8.

J K Pinter, Dr.Tech.

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Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

Introduction

Pinter Consulting of Calgary, Alberta practices Mathematics, promotes clear thinking and offers Consultations, Tutorials and Seminars in Mathematics.

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Chapter 8

Proceedings

8.1 Summary of Current Report

- **Private study for professional development:**
- Records of activities at Pinter Consulting
- Collection of problems with our own solutions .
- GRE Mathematics Test (Resource: www.ets.org/gre)
- In compliance with Terms of Use ”fair use for educational purposes”
- Continuous improvement, corrections and last revisionJuly 5, 2015.

8.2 Solutions 1-11.

- Mathematics Test (GRE)
- *Spartan Old School Tutorials*
- Last revision July 5, 2015

Problems

1. In the xy-plane, the curve with parametric equations $x = \cos t$ and $y = \sin t$, $0 \leq t \leq \pi$, has length π .

Solution: Consider the standard Cartesian coordinate system. Let the unit vector have one end at the origin, and let it rotate counterclockwise. The projection onto the $x - axis$ is $\cos t$, and on the $y - axis$ is $\sin t$. In one revolution, $t = [0, 2\pi]$, the unit vector traces the unit circle, which has perimeter 2π. Therefore the arc that belongs to $0 \leq t \leq \pi$ is a semicirle, with arclength π.

2. The equation of the line tangent to the graph of $y = e^x + x$ at $x = 0$ is $y = 2x + 1$.

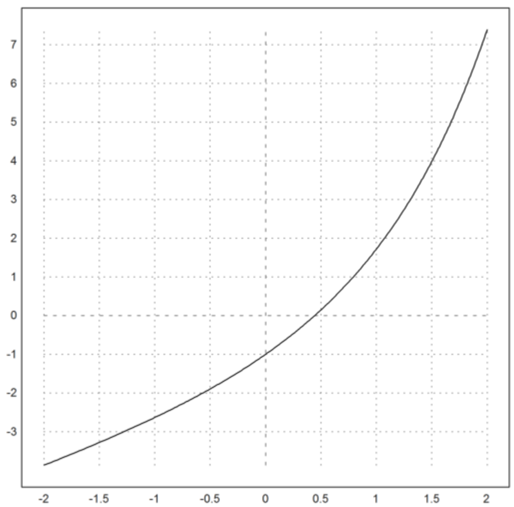
Solution: $y(0) = e^0 + 0 = 1$, $y'(x) = e^x + 1$, $y'(0) = 1 + 1 = 2$ therefore tangent passes through (0,1) and has slope of 2, $y = 1 + 2(x - 0) = 2x + 1$.

3. If V and W are 2-dimensional subspaces of R^4, the possible dimensions of the subspace $V \cap W$ are 0, 1, and 2.

Solution: If $V \cap W = \emptyset$, then subspace $V \cap W$ has zero dimension. If $V \cap W$ is not empty then there is at least one non-zero vector u_1 in it. $V \cap W$ is subspce so it contains all linear combinations of u_1. If u_1 actually spans $V \cap W$ then $V \cap W$ has one dimension. If u_1 does not span $V \cap W$ then there is another non-zero vector u_2, such that $\{u_1, u_2\}$ are linearly independent vectors in $V \cap W$ and $\dim(V \cap W) = 2$. Further, $V \cap W \subseteq V$ as well as $V \cap W \subseteq W$ and the dimension of the intersection cannot exceed the dimensions of V or W, so $\dim(V \cap W) \leq 2$.

4. Let k be the number of real solutions of the equation $e^x + x - 2 = 0$ in the interval $[0, 1]$, and let n be the number of real solutions that are not in $[0, 1]$. Show that $k = 1$ and $n = 0$.

Solution: First, note that $f(x) = e^x + x - 2$ is strictly monotone increasing, $f'(x) = e^x + 1 > 0$ for $\forall x$. Second, if $x < 0$ then $f(x) < -1$; third, $f(0) = e^0 + 0 - 2 = -1$; fourth, $f(1) = e^1 - 1 > 0$; so there is root in $[0, 1]$ and finally if $x > 1$ then $f(x) > f(1)$. Approximately, $f(0.442854) = 0.0$.



5. Suppose b is a real number and $f(x) = 3x^2 + bx + 12$ defines a function such that $f(2) = 0$, then f(5)=27.

Solution: $f(2) = 3 * 2^2 + b * 2 + 12 = 0$ implies $24 + 2b = 0$ hence $b = -12$. $f(5) = 3 * 25 - 12 * 5 + 12 = 27$.

6. Given circle $x^2 + y^2 = r^2$ and parabola $x^2 = y + 4$. Determine the number of points of intersection when $r = 1, 2, 3, 4, 5$.

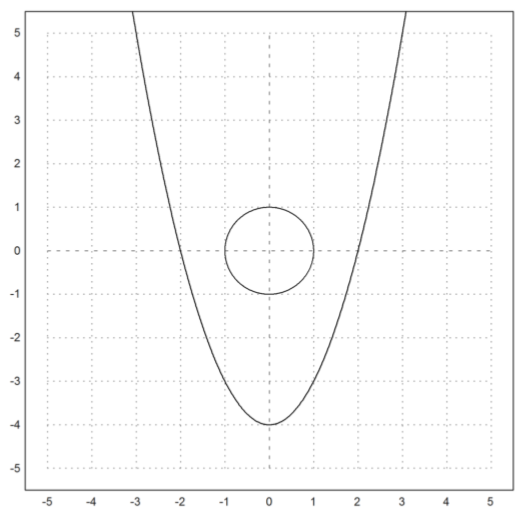
Solution: Solve the system of non-linear equtions:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ y + 4 &= x^2. \end{aligned}$$

Substitution of the second equation in the first yields a quadratic equation in y.

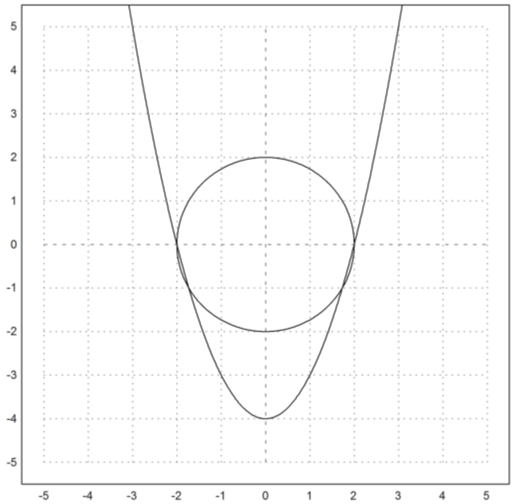
$$\begin{aligned} y + 4 + y^2 &= r^2 \\ y^2 + y + (4 - r^2) &= 0 \\ y_{1,2} &= \frac{-1 \pm \sqrt{1^2 - 4(4 - r^2)}}{2} \\ y_{1,2} &= \frac{-1 \pm \sqrt{4r^2 - 15}}{2} \\ \Delta &= \sqrt{4r^2 - 15} \end{aligned}$$

Case A: $r = 1$, $\Delta < 0$, no real solution to quadratic equation, no points of intersection.

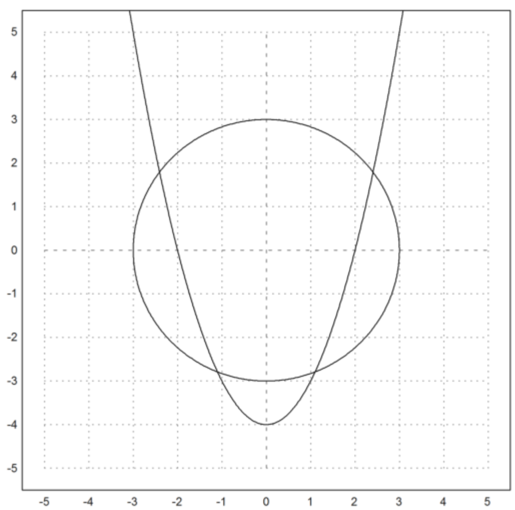


Case B: $r = 2$, $\Delta = 1 > 0$, two real solutions to quadratic equation, four points of intersection. Notice that both $y = 0$ and $y = -1$ are solutions:

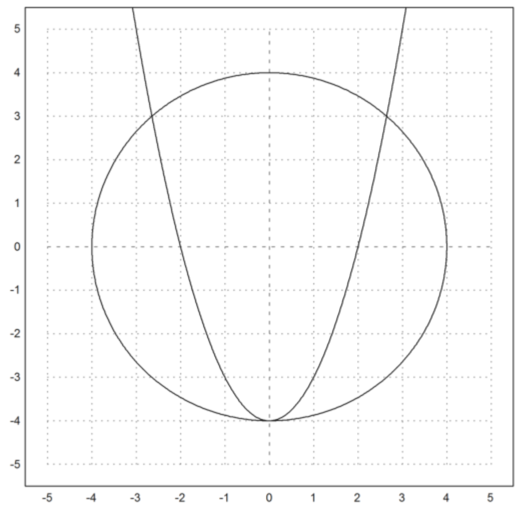
$$\begin{aligned} x^2 + y^2 &= 4 \\ y + 4 &= x^2. \end{aligned}$$



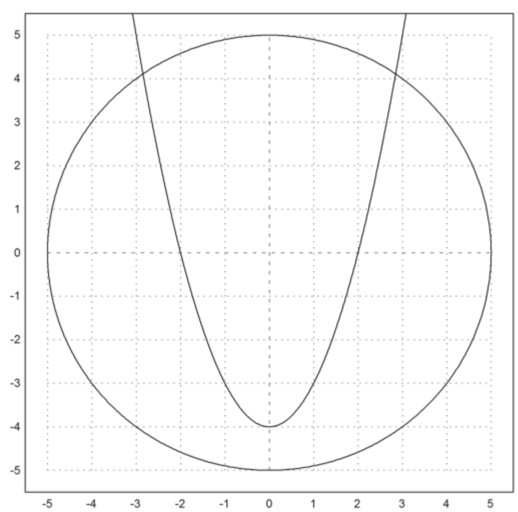
Case C: $r = 3$, $\Delta > 0$, two real solutions to quadratic equation, four points of intersection.



Case D: $r = 4$, $\Delta > 0$, two real solutions to quadratic equation, three points of intersection. There is a tangential contact at $y = -4$.

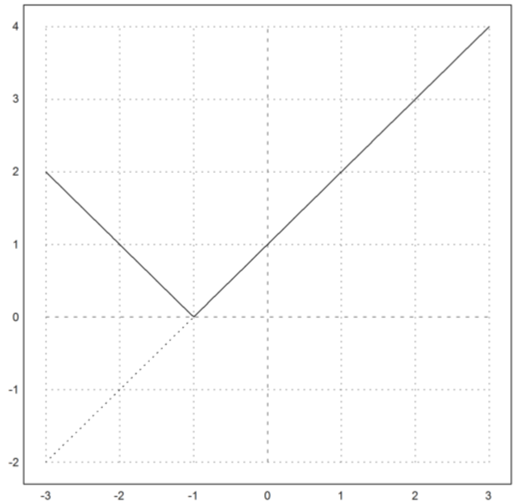


Case E: $r = 5$, $\Delta > 0$, two real solutions to quadratic equation, two points of intersection. The vertex of the parabola is inside the circle.



7.

$$\int_{-3}^{3} |x+1| dx = 10.$$



Calculate the areas of two triangles.

$$\frac{2*2}{2} + \frac{4*4}{2} = 10.$$

8. The greatest area of a triangular region with one vertex at the center of the cirle of radius 1 and the other two vertices on the circle is $\frac{1}{2}$.

Solution: Consider $\triangle OPQ$, where point O is the center and points P, Q are on the circle, $P \neq Q$. $\triangle OPQ$ is an isosceles triangle, for $OP = OQ = 1$ by construction. Let α denote the angle at O. The bisector of the vertex angle at O is a median to the base. Call the median M. Segment OM is perpendicular to the base. Then the altitude of the triangle $\triangle OPQ$ is OM and is equal to $\cos(\frac{\alpha}{2})$, by inspection. Moreover,the base of the triangle is $2\sin(\frac{\alpha}{2})$. Therefore the area is

$$\frac{1}{2}(2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2})) = \sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2}) = \frac{1}{2}\sin(\alpha),$$

by the half-angle formula

$$2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2}) = \sin(\alpha).$$

Trigonometric function $\sin(\alpha)$ has a maximum at $\frac{\pi}{2}$, $\sin(\frac{\pi}{2}) = 1$.

9.

$$J = \int_0^1 \sqrt{1-x^4}dx$$
$$K = \int_0^1 \sqrt{1+x^4}dx$$
$$L = \int_0^1 \sqrt{1-x^8}dx$$

$$J < L < 1 < K$$

Solution: First, we look at the intenrands in $(0,1)$, then augment the results at the endpoints. In $(0,1)$ every integrand is positive, and

$$1 + x^4 > 1$$
$$1 - x^4 < 1$$
$$1 - x^8 < 1.$$

For numbers in $(0,1)$ $x^4 > x^8$, hence

$$1 - x^4 < 1 - x^8$$

Therefore

$$\begin{aligned} 1 - x^4 < 1 - x^8 &< 1 < 1 + x^4 \\ \sqrt{1-x^4} < \sqrt{1-x^8} &< \sqrt{1} < \sqrt{1+x^4} \end{aligned}$$

in the open interval $(0,1)$ taking positive square roots. The square root operation is monotone. Further, at $x = 0$

$$1 - 0 \leq 1 - 0 \leq 1 \leq 1 + 0$$

and at $x = 1$

$$0 \leq 0 \leq 1 < 2$$

These values at the endpoints do not change the facts that the functions are integrable or the value of the integrals

$$\begin{aligned} \int_0^1 \sqrt{1-x^4}dx < \int_0^1 \sqrt{1-x^8}dx &< \int_0^1 \sqrt{1}dx < \int_0^1 \sqrt{1+x^4}dx \\ \int_0^1 \sqrt{1-x^4}dx < \int_0^1 \sqrt{1-x^8}dx &< 1 < \int_0^1 \sqrt{1+x^4}dx. \end{aligned}$$

10. Let g be a function on $[0, a], a > 5$ whose derivative g' is continuous and

$$\begin{array}{rcl} g'(x) & > & 0, \quad 0 \leq x < 2 \\ g'(x) & = & 0, \quad x = 2 \\ g'(x) & < & 0, \quad 2 \leq x < 5 \\ g'(x) & = & 0, \quad x = 5 \\ g'(x) & > & 0, \quad 5 \leq x \leq a. \end{array}$$

Then g has a maximum at $x = 2$.

11. Approximate $(266)^{\frac{3}{2}}$.

$$\begin{array}{rcl} 16^2 = 256 & < & 266 < 289 = 17^2 \\ 16 & < & \sqrt{266} < 17 \\ 1.2^2 = 1.44 & < & 1.5 < 1.5625 = 1.25^2 \\ 1.2 & < & \sqrt{1.5} < 1.25 \\ 1.2 * (16)^3 & < & \sqrt{1.5}(266)^{\frac{3}{2}} < 1.25 * (17)^3 \\ 4915.2 & < & \sqrt{1.5}(266)^{\frac{3}{2}} < 6141.25 \end{array}$$

$$\sqrt{1.5}(266)^{\frac{3}{2}} = 5313.35$$

8.3 Solutions 12-22.

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Problems

12. Let A be a 2×2 matrix for which there is a constant k such that the sum of the entries in each row and each column is k. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ must be an eigenvector of A. Further, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ may or may not be eigenvectors of A.

Solution: Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$a + b = c + d = a + c = b + d = k.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This shows that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. Further

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \; a = k, \; c = 0 \Rightarrow \; d = k, \; b = 0.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \; b = 0, \; d = 1 \Rightarrow \; a = k, \; c = 0.$$

Therefore $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, a special case of A.

13. A total of L feet of fencing is to form three sides of a level rectangular yard. The maximum possible area of the yard is $\frac{L^2}{8}$.

Solution: Variables are x and y, L is constant. Let A denote the area.

$$2x + y = L, \; xy = A$$

Maximize A, subject to constraint.

$$x(L - 2x) = A$$

$$Lx - 2x^2 = A$$

$$\frac{dA}{dx} = L - 4x = 0$$

$$x = \frac{L}{4}, \; y = \frac{L}{2}$$

$$\max A = \frac{L^2}{8}.$$

14. The unit digit in the standard decimal expansion of 7^{25} is 7.

Solution: The unit digit is the remainder in modulo 10. Apply Euler's theorem,

$$a^{\phi(m)} \equiv 1, \; (mod\ m\ ), (a, m) = 1,$$

with $a = 7$ and $m = 10$. 7 and 10 are relative primes, so $(a, m) = 1$ is fulfilled. $\phi(10) = 4$ because in the complete set of residues $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ there are 4 relative primes to $10 = 2 * 5$, $\{1, 3, 7, 9\}$.

$$7^4 = 49 \times 49 = 2401$$

$$7^4 \equiv 1, \quad (mod\ 10\ ).$$

$$(7^4)^6 \equiv 1^6, \quad (mod\ 10\ ).$$

Finally,

$$7 * 7^{24} \equiv 7 * 1, \quad (mod\ 10\ ).$$

Demonstration: Let us examine Euler's Theorem in more detail. Assuming the notions of divisibility of positive integers also known as natural numbers and prime numbers, recall that $(a, b) = H.C.F.$, the highest common factor (divisor) of integers a and b, and $\phi(a)$ is the number of numbers up to a that are relative prime to a or $(a, b) = H.C.F. = 1$. The reduced system of residues in modulo a consists of numbers less than a that are relative prime to a.

Next, consider the reduced system of residues in modulo 10 and multiply each number by 7 which is relative prime to 10:

$$\{1, 3, 7, 9\} \times 7 = \{7, 21, 49, 63\} \equiv \{7, 1, 9, 3\} \ (\textit{mod 10})$$

We obtain the same set of numbers, in different order. See what happens when we multiply these numbers together

$$1 * 3 * 7 * 9 \equiv (7 * 1) * (7 * 3) * (7 * 7) * (7 * 9) = 7^4 * (1 * 3 * 7 * 9)$$

Simplify both sides by $(1 * 3 * 7 * 9)$

$$1 \equiv 7^4 \ (\textit{mod 10}).$$

Congruences (of the same modulus) can be added, subtracted or multiplied together just like equations

$$(1)^6 \equiv (7^4)^6 \ (\textit{mod 10})$$

$$1 \equiv 7^{24} \ (\textit{mod 10}).$$

Finally, multiply both sides by 7

$$7 \equiv 7^{25} \ (\textit{mod 10}).$$

15. Let f be a continuous real-valued function defined on the closed interval $[-2, 3]$. Then

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

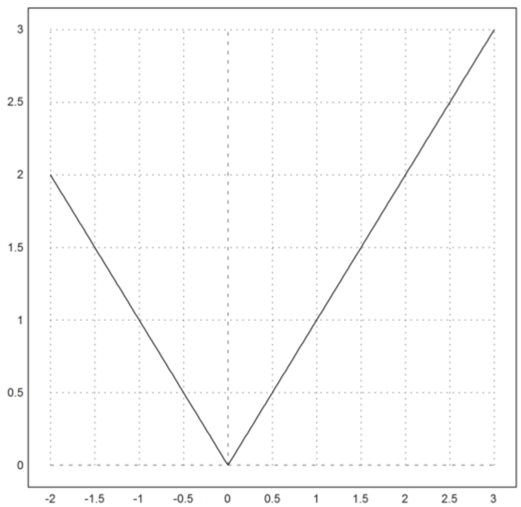
may not exist.

Solution: Note that f is bounded, takes on its maximum and minimum and all values between. It is uniformly continuous and it behaves very well under integration. But for differentiation at x

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$$

is required. So if we set $f(x) = |x|$, $f(x)$ is not differentiable at $x = 0$ because

$$\lim_{h \to +0} \frac{f(h) - f(0)}{h} \neq \lim_{h \to -0} \frac{f(h) - f(0)}{h}.$$



16. The volume V_x of the solid formed by revolving about the x-axis the region in the first quadrant of the xy-plane bounded by the coordinate axes and the graph of the equation $y = \dfrac{1}{\sqrt{1+x^2}}$ is $\dfrac{\pi^2}{2}$.

Solution: Recall that the volume V_x of the solid is approximated by disks:

$$V_x \approx \sum \Delta V_x = (y(x)^2) * \pi * \Delta x,$$

after the limiting process this turns into an integral

$$V_x = \pi \int_0^\infty y^2 dx.$$

Now we have to show that the improper integral exists. First we find the indefinite integral, then apply a limiting process on the upper limit of integration.

Claim:

$$\int \frac{dx}{1+x^2} = \arctan(x)$$

Proof of Claim:

$$y = \arctan(x)$$

$$\tan(y) = x$$

$$\frac{d}{dx}\tan(y) = \frac{d}{dx}x = 1$$

$$\frac{d}{dx}\tan(y) = (1+\tan^2(y))\frac{dy}{dx} = 1.$$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

$$dy = \frac{dx}{1+x^2}$$

$$\int dy = \int \frac{dx}{1+x^2}$$

$$y = \int \frac{dx}{1+x^2}$$

$$\int \frac{dx}{1+x^2} = \arctan(x).$$

End of Proof of Claim.

$$V_x = \lim_{M \to \infty} \pi \int_0^M \frac{dx}{1+x^2} = \lim_{M \to \infty} \pi\left(\arctan(M) - \arctan(0)\right) = \frac{\pi^2}{2}.$$

17. The polynomial $2x^5 + 8x - 7$ has one real root.

Proof: The polynomial is strictly monotone increasing for $-\infty < x < \infty$ because its derivative $10x^4 + 8$ is positive.

$$\lim_{x \to -\infty} 2x^5 + 8x - 7 = -\infty$$

$$\lim_{x \to +\infty} 2x^5 + 8x - 7 = +\infty$$

$$2 * 0^5 + 8 * 0 - 7 < 0$$

$$2 * 1^5 + 8 * 1 - 7 > 0$$

Therefore there is a real root in $[0, 1]$, and there are no other real roots.

Of course the polynomial $2x^5 + 8x - 7$ has 5 roots, guarranteed by the Fundamental Theorem of Algebra. The complex roots come in pairs, if z is a root then so is its complex conjugate $\overline{z}$. There are other methods to determine the number of real roots, but the above examination seems to be straightforward and simple.

18. Let V be the real vector space of all real 2×3 matrices, and let W be the real vector space of all real 4×1 column vectors. If T is linear transformation from V onto W, the dimension of the subspace $\{v \epsilon V : T(v) = 0\}$ is 2.

Demonstration: The result is the consequence of the dimension theorem:

$$\dim(\text{im } T) + \dim(\ker T) = \dim(V).$$

We assume the basic notions of linear algebra (linear combination, linear independence, basis, dimension etc.) and attempt to explain the dimension theorem. First, we state that

$$\dim V = 6, \quad \dim W = 4,$$

a possible basis for V is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and for W is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The null-space or kernel of a linear transformation T from V onto W

$$\ker T = \{v\epsilon V : T(v) = 0\},$$

is a subspace of V

$$v\epsilon V : T(v) = 0;\ u\epsilon V : T(u) = 0 \Rightarrow T(v + cu) = T(v) + cT(u) = 0$$

thus $v + cu$ is in $\ker T$, when c is a real coefficient $c \neq 0$.

The range or the image of the linear transformation T

$$\text{im } T = \{y\epsilon W : \exists x\epsilon V,\ y = T(x)\},$$

is a subspace of W for if $y_1 \epsilon W$, and $y_2 \epsilon W$ then $\exists x_1 \epsilon V$ and and $\exists x_2 \epsilon V$ such that $y_1 = T(x_1)$ and $y_2 = T(x_2)$. By the linearity of T, $y_1 + cy_2 = T(x_1 + cx_2)$, $c \neq 0$, thus linear combinations of y_1 and y_2 are in W.

It is important to note that $\ker T$ and im T are in *two different* vector spaces, $\ker T$ is in V and under linear transformation T it goes to the zero element of W and the rest of V goes to im T in W.

Next, we build a basis $\{a_1, \dots a_r, a_{r+1}, \dots a_n\}$ in V, the first r vectors span the kernel and the remaining $n - r$ vectors span the rest of V. Write

$$\{b_1, \dots b_r, b_{r+1}, \dots b_n\}$$

for the images of basis vectors. Obviously

$$T(a_i) = b_i = 0,\ i = 1 \dots r.$$

Consider a linear combination of basis vectors $\{a_{r+1}, \dots a_n\}$. (We try to work up a contradiction here.) If there are coefficients c_k, not all zero, such that

$$T(\sum_{r+1}^{n} c_k a_k) = \sum_{r+1}^{n} c_k b_k = 0$$

then $\sum_{r+1}^{n} c_k a_k$ is in $\ker T$ and as such can be expressed by a linear combination of basis vectors $\{a_1, \dots a_r\}$ of $\ker T$:

$$\sum_{r+1}^{n} c_k a_k = \sum_{1}^{r} c_i a_i$$

which means

$$c_1 a_1 \dots c_r a_r - c_{r+1} a_{r+1} \dots - c_n a_n = 0$$

for certain $c_j, j = 1 \dots n$ not all zero. This is absurd, because basis vectors are linearly independent. Therefore basis vectors

$$\{a_{r+1}, \dots, a_n\}$$

map into linearly independent vectors

$$\{T(a_{r+1}), \dots T(a_n)\} \equiv \{b_{r+1}, \dots b_n\}.$$

in W. Furthermore, transformation T is *onto*

$$W = \text{im } T$$

which makes $\{b_{r+1}, \dots b_n\}$ a basis of W. So in our case $n = 6$, $\{a_1, a_2\}$ spans the kernel of T and

$$\{T(a_3), \dots T(a_6)\} \equiv \{b_3, \dots b_6\}$$

spans the im T.

$$\dim(\ker T) = 2.$$

19. Let f and g be twice-differentiable real-valued functions defined on R. If $f'(x) > g'(x)$ for all $x > 0$, then $f(x) - f(0) > g(x) - g(0)$.

Proof: Clearly

$$f(x) = \int f'(x)dx, \quad g(x) = \int g'(x)dx$$

indefinite integrals exist by construction. Let a be a positive number, then

$$\int_0^a f'(x)dx = f(a) - f(0), \quad \int_0^a g'(x)dx = g(a) - g(0)$$

by the *Fundamental Theorem of Calculus* .

Moreover, $f'(x) > g'(x)$ for all $x > 0$ implies that $\exists\varepsilon > 0$, $f'(x) - g'(x) \geq \varepsilon$ and

$$I = \int_0^a (f'(x) - g'(x))dx \geq \varepsilon \times a > 0$$

$$I = (f(a) - f(0)) - (g(a) - g(0)) \,.$$

Therefore

$$(f(a) - f(0)) - (g(a) - g(0)) > 0$$

$$f(a) - f(0) > g(a) - g(0).$$

Since the choice of positive a was arbitrary we can write

$$f(x) - f(0) > g(x) - g(0).$$

20. Let f be the function defined on the real line by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is rational} \\ \dfrac{x}{3} & \text{if } x \text{ is irrational} \end{cases}$$

If D is the set of points of discontinuity of f, then D is the set of nonzero real numbers.

Demonstration: It is enough to show that f is continuous at 0 and no other point. Let us recall the standard definitions.

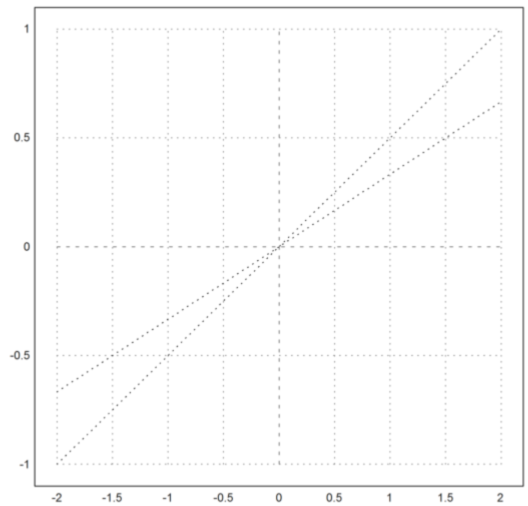
i) The function f is continuous at p if

$$\lim_{x \to p} f(x) = f(p)$$

ii) The function f is continuous at p if for every $\varepsilon > 0$ there exists a δ such that

$$|f(x) - f(p)| < \varepsilon \text{ whenever } |x - p| < \delta.$$

Next, look at the misleading but informative picture of $f(x)$:



The dots, of course do not represent rational or irrational numbers, respectively; they only attempt to depict the non-continuous nature of the two branches of of $f(x)$. So if $x \neq 0$ then

$$\lim_{rational\ x \to p} f(x) = \frac{x}{2}$$

$$\lim_{irrational\ x \to p} f(x) = \frac{x}{3}.$$

Clearly, $\frac{x}{2} \neq \frac{x}{3}$, for $x \neq 0$, therefore $f(x)$ is discontinuous if $x \neq 0$. On the other hand, $f(x)$ is continuous at $x = 0$, check for example Definition ii).

Remark: The function in this exercise is a variant of the famous *Dirichlet-function*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

which can be expressed as

$$\lim_{\nu \to \infty} \left(\lim_{k \to \infty} \left(\cos(\nu! \pi x)^{2k} \right) \right).$$

Let x be a *rational number* . Then $x = \frac{p}{q}$ where p and q are integers, and relative primes. When ν is greater than q

$$(\nu!\pi x) = (\nu!\pi \frac{p}{q}) = M\pi$$

for some integer M.

$$\cos(\nu!\pi x) = \cos(M\pi) = \pm 1$$

$$\cos(\nu!\pi x)^{2k} = 1$$

Now let x be an *irrational number* .

$$\cos(\nu!\pi x) = \cos(X\pi) = y$$

where X is *not an integer* and $-1 < y < 1$.

$$\lim_{k \to \infty} y^{2k} = 0.$$

Dirichlet function is used in counterexamples, for example this function is not *Riemann-integrable.*

21. Let P_1 be the set of all primes, $\{2, 3, 5, 7, \ldots\}$, and for each integer n, let P_n be the set of the set of all prime multiples of n, $\{2n, 3n, 5n, 7n, \ldots\}$ then

i) $P_1 \cap P_{23} = \emptyset$

$$P_1 = \{2, 3, 5, 7, \ldots\}$$

$$P_{23} = \{2 * 23, 3 * 23, 5 * 23, 7 * 23, \ldots\}$$

P_1 contains prime numbers. P_{23} is a set of composite nunbers. No match.

ii) $P_7 \cap P_{21} = \emptyset$

$$P_7 = \{2 * 7, 3 * 7, 5 * 7, 7 * 7, \ldots\}$$

$$P_{21} = \{2 * 21, 3 * 21, 5 * 21, 7 * 21, \ldots\} =$$

$$\{(2 * 3) * 7, (3 * 3) * 7, (5 * 3) * 7, (7 * 3) * 7, \ldots\}.$$

The typical element of P_7 is $p \times 7$, where p is prime, The typical element of P_{21} is $m \times 7$, where m is composite. No match.

iii) $P_{12} \cap P_{20} \neq \emptyset$

$$P_{12} = \{12 * 2, 12 * 3, 12 * 5, 12 * 7, \ldots\}$$

$$P_{20} = \{2 * 20, 3 * 20, 5 * 20, 7 * 20, \ldots\}$$

$$12 * 5 = 5 * 20.$$

iv) $P_{20} \cap P_{24} = \emptyset$

$$P_{20} = \{2 * 20, 3 * 20, 5 * 20, 7 * 20, \ldots\}$$

$$P_{24} = \{2 * 24, 3 * 24, 5 * 24, 7 * 24, \ldots\}$$

$$20 = 5 * 2^2$$

P_{20} has multiples ot 5. P_{24} has only one multiple of 5, 5*24 = 120 = 6* 20 not an element of P_{20}.

v) $P_5 \cap P_{25} = \emptyset$

$$P_5 = \{2 * 5, 3 * 5, 5 * 5, 7 * 5 \ldots\}$$

$$P_{25} = \{2 * 25, 3 * 25, 5 * 25, 7 * 25 \ldots\}$$

P_5 has one multiple of 25, $5 * 5$, P_{25} has multiples of 25, the smallest is $2 * 25$, no match.

22. Let $C(R)$ be the collection of all continuous functions from R to R. Then $C(R)$ is a real vector space with pointwise addition and scalar multiplication defined by

$$(f + g)(x) = f(x) + g(x)$$

$$(rf)(x) = rf(x)$$

for all $f, g \epsilon C(R)$ and all $r, x \epsilon R$.

i) f is twice and differentiable and

$$\{f"(x) - 2f'(x) + 3f(x) = 0, \quad \forall x\}$$

is a subspace:

$$(f_1"(x) - 2f_1'(x) + 3f_1(x)) + (f_2"(x) - 2f_2'(x) + 3f_2(x)) = 0$$

$$r\,(f_1"(x) - 2f_1'(x) + 3f_1(x)) = 0$$

ii) g is twice and differentiable and

$$g"(x) = 3g'(x)$$

is a subspace. Write $g"(x) - 3g'(x) = 0$ and apply the definitions.
iii) h is twice and differentiable and

$$h"(x) = h(x) + 1$$

is not a subspace. Write $h"(x) - h(x) = 1$

$$r(h"(x) - h(x)) = rh"(x) - rh(x) = r \neq 1$$

is not in the prescibed form.

8.4 Solutions 23-33.

- Mathematics Test (GRE)
- *Spartan Old School Tutorials*
- Last revision July 5, 2015

Problems

23. For what value of b is the line $y = 10x$ tangent to the curve $y = \exp(bx)$.

Solution: Write $f(x) = 10x$ and $g(x) = \exp(bx)$. The curves $f(x), g(x)$ have a tangential contact at point x if

$$f(x) = g(x), \quad f'(x) = g'(x).$$

$$\begin{aligned} 10x &= \exp(bx) \\ 10 &= b\exp(bx) \\ xb\exp(bx) &= \exp(bx) \\ xb &= 1 \\ 10/b &= \exp(1) \\ 10/\exp(1) &= b. \end{aligned}$$

24. Let h be the function defined by

$$h(x) = \int_0^{x^2} \exp(x+t)dt$$

for all real numbers x. Then $h'(1) = 3\exp(2) - \exp(1)$.

Proof: *Leibniz-rule* of differentiation under the integral sign says

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t,x)dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(t,x)dt + f(b,x)\frac{db}{dx} - f(a,x)\frac{da}{dx}.$$

$$h'(x) = \frac{d}{dx}\int_0^{x^2} \exp(x+t)dt = \int_0^{x^2} \frac{\partial}{\partial x}\exp(x+t)dt + exp(x^2+x)2x - exp(0+x) * 0$$

$$\int_0^{x^2} \frac{\partial}{\partial x} \exp(x+t)dt = \int_0^{x^2} \exp(x+t)dt = \exp(x+x^2) - \exp(x+0)$$
$$h'(x) = \exp(x+x^2) - \exp(x+0) + \exp(x^2+x)2x$$
$$h'(1) = \exp(2) - \exp(1) + 2\exp(2) = 3\exp(2) - \exp(1).$$

25. Let $\{a_n\}_{n=1}^{\infty}$ be defined recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{(n+2)}{n} a_n$$

then $a_{30} = (15)(31)$.

Demonstration: Due to the recursive nature of this problem we will find the solution by looking for a pattern:

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= \frac{(1+2)}{1} a_1 = 2 \\
a_3 &= \frac{(1+3)}{2} a_2 = 4 \\
&\vdots \\
a_{n-1} &= \frac{(n)}{n-2} a_{n-2} \\
a_n &= \frac{(n+1)}{n-1} a_{n-1} \\
a_{n+1} &= \frac{(n+2)}{n} a_n.
\end{aligned}$$

In reverse order:

$$\begin{aligned}
a_{n+1} &= \frac{(n+2)}{n} a_n. \\
a_{n+1} &= \frac{(n+2)}{n} \frac{(n+1)}{n-1} a_{n-1}. \\
a_{n+1} &= \frac{(n+2)}{n} \frac{(n+1)}{n-1} \frac{(n)}{n-2} a_{n-2}. \\
a_{n+1} &= \frac{(n+2)}{n} \frac{(n+1)}{n-1} \frac{(n)}{n-2} \cdots \frac{(3)}{1} a_1
\end{aligned}$$

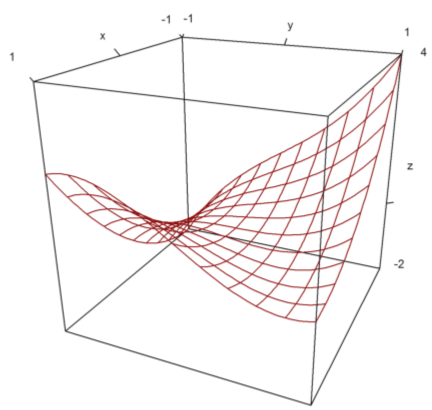
In the numerator we have $(n+2)(n+1)(n)(n-1)(n-2)\ldots 3$, a descending sequence of integers, n terms; in the denominator $(n)(n-1)(n-2)(n-3)\ldots 1$, another descending sequence of integers, also n terms and $a_1 = 1$. Therefore

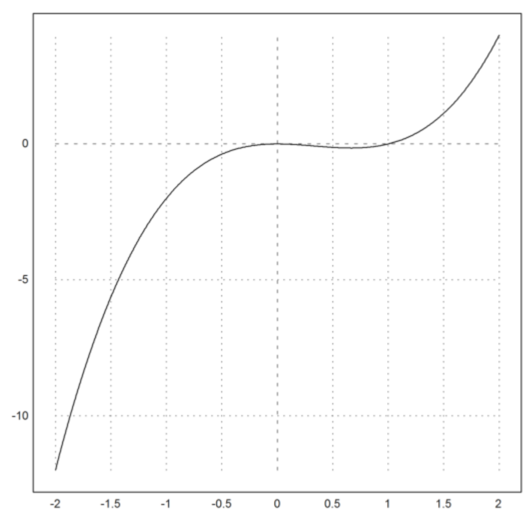
$$a_{n+1} = \frac{(n+2)(n+1)}{2}.$$

$$a_{30} = \frac{(31)(30)}{2} = (31)(15).$$

26. Let $f(x,y) = x^2 - 2xy + y^3$ for all real x and y. Then f has all of its relative extrema on the line $x = y$.

Discussion: This is a very instructive exercise to show that naive use of computer graphics can lead to errors. Let us display $f(x,y)$.





The surface of $f(x,y)$ is ambiguous, the cross section at along the line $x-y=0$ might suggest a very gentle local maximum at $(0,0)$ and a minimum at $(\frac{2}{3}, \frac{2}{3})$.

However if f is a real-valued function defined on a domain G and P_0 is a point in its interior and there exists a neighborhood U of P_0 such that for every point $P(\neq P_0)$ we have $f(P) \geq f(P_0)$ then we say that f has a *relative minimum* at P_0. *Relative maximum* is defined similarly with $f(P) \leq f(P_0)$. *Relative extremum* is either a relative minimum or a relative maximum. The necessary condition for a twice differentiable function $f(x,y)$ to have a *relative extremum* is

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

In our particular case

$$\frac{\partial f}{\partial x} = 2x - 2y = 0 \rightarrow \ x = y$$

$$\frac{\partial f}{\partial y} = -2x + 3y^2 = 0$$

$$(2y - 2y) + (-2y + 3y^2) = 3y^2 - 2y = 0$$

$$y = 0, \ y = \frac{2}{3} \ \rightarrow \ (0,0); \ (\frac{2}{3}, \frac{2}{3}).$$

The vanishing of the first derivatives does not quarantee a relative extremum so we have to run the Second Derivative Test.

$$J_f(x) = \begin{bmatrix} f_{xx} & f_{xy} \\ fyx & fyy \end{bmatrix}$$

is the $Jacobian(determinant)$ of f at x.

$$J_f(x) = \begin{bmatrix} 2 & -2 \\ -2 & 6y \end{bmatrix} = (2 * 6y) - [(-2) * (-2)] = 12y - 4.$$

At $(0,0)$ there is a *saddle point* not a relative extremum

$$J_f(0) = 12 * 0 - 4 < 0.$$

Indeed,

$$\lim_{y \to +0} f(0,y) = \lim_{y \to +0} y^3 > 0$$

$$\lim_{y \to -0} f(0,y) = \lim_{y \to -0} y^3 < 0.$$

At $(\frac{2}{3}, \frac{2}{3})$ there is a a relative extremum (minimum)

$$J_f(\frac{2}{3}) = 12 * \frac{2}{3} - 4 > 0, f_{xx} = 2 > 0.$$

27. Consider two planes $x + 3y - 2z = 7$ and $2x + y - 3z = 0$ in R^3. Then the set $L = \{(x, y, z) : x = 7t,\ y = 3 + t,\ z = 1 + 5t,\ t\epsilon R\}$ is the intersection of the plains. The set $M = \{(x, y, z) : x = t,\ y = 3t,\ z = 7 - 2t,\ t\epsilon R\}$ is not an intersection.

Proof: Sustitution of L yields equlities.

$$(7t) + 3(3 + t) - 2(1 + 5t) = 7t + 9 + 3t - 2 - 10t = 7$$

$$2x + y - 3z = 2(7t) + (3 + t) - 3(1 + 5t) = 0$$

Sustitution of M does not yield equlities.

$$t + 3 * 3t - 2(7 - 2t) = 14t - 14 \neq 7$$

Note that the question was to choose one set out of 5 sets. The empty set can be eliminated because the planes are not parallel. Further one single point cannot be the intersection. If the intersection of two planes is a plane then the planes are identical.

Remark: The standard method to find the intersection of two planes by linear algebra is as follows:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & -3 \end{bmatrix}; \quad AH = \begin{bmatrix} 1 & 3 & -2 & 7 \\ 2 & 1 & -3 & 0 \end{bmatrix}$$

Coefficient matrix A has *rank* 2, because there is minor of order 2 that does not vanish

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \neq 0.$$

Augmented matrix AH is of *rank* 2 as well, for the same reason. Therefore the system of linear equations is *consistent* . Number of unknowns less rank is one, thus there is one parameter in the solution. Let us find the solution by elementary row operations.

$$\begin{bmatrix} 1 & 3 & -2 & 7 \\ 2 & 1 & -3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -2 & 7 \\ 0 & -5 & 1 & -14 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -2 & 7 \\ 0 & 1 & -0.2 & 2.8 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & -1.4 & -1.4 \\ 0 & 1 & -0.2 & 2.8 \end{bmatrix}.$$

$$x - 1.4z = -1.4$$
$$y - 0.2z = 2.8$$

Set $z = u$. Then

$$x = -1.4 + 1.4u$$
$$y = 2.8 + 0.2u$$
$$L' = \{(x, y, z) : x = -1.4 + 1.4u,\ y = 2.8 + 0.2u,\ z = u,\ u \epsilon R\}$$

is the intersection of the two planes, in different parametric representation. The intersection of two planes can be found by analytic geometry too, using the *cross-products* of the normal vectors to each plane.

28. Given an undirected, cyclic, finite graph Γ with six vertices and nine edges:

$$\{A, B, C, D, E, F\},\ \{AD, DB, BE, EC, CF, FA, DE, EF, FD\}.$$

One obvious cycle is

$$\{AD, DB, BE, EC, CF, FA\},$$

(actually a *Hamiltonian cycle*). A spanning tree Γ' is a connected subgraph of Γ with the same six vertices and no cycles, or put it differently, a finite graph Γ' is a tree of Γ if and only if for any two of its vertices there exists exactly one path which joins them. Every connected finite graph has a tree. Any finite connected graph with n vertices has at least $n - 1$ edges. A tree is a minimal graph in the sense that every tree on n vertices has exactly $n - 1$ edges. Therefore Γ' has exactly 5 edges and $9 - 5 = 4$ edges have to be removed to make Γ'.

$$\Gamma' = \{AD, DB, BE, EC, CF\},$$

29. For all positive functions f and g of the real variable x, let $\sim$ be a relation defined by

$$f \sim g \iff \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

Show that

$$\exp(f) \sim \exp(g)$$

is *not* a consequence of $f \sim g$.

Proof: Set

$$f(x) = x + 1, \ g(x) = x.$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x+1}{x} = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right) = 1 \ \Rightarrow f \sim g.$$

$$\lim_{x \to \infty} \frac{\exp(f(x))}{\exp(g(x))} = \lim_{x \to \infty} \exp(f(x) - g(x)) = \exp(1) \neq 1.$$

Therefore $\exp(f(x)), \ \exp(g(x))$ are not in relation $\sim$.

30. Let f be a function from a set X to a set Y. Consider the following statements.

P: For each $x \epsilon X \ \exists : \ y \epsilon Y$ such that $f(x) = y$.

Q: For each $y \epsilon Y \ \exists : \ x \epsilon X$ such that $f(x) = y$.

R: $\exists x_1, x_2 \epsilon X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Then the negation of the statement " f is one-to-one and onto Y " is R and not Q.

Discussion: Statement P signifies that set X is mapped to set Y. Statement Q means that every element y in Y has a *pre-image* or to put it differently $Y = \text{im } f$, f is "onto" Y. The third statement, R, is the *negation* that f is one-to one. Recall further

$$\neg(A \wedge B) = \neg A \vee \neg B.$$

Therefore R or not Q negates the statement "f is one-to-one and onto Y".

31. Inspection and qualitative picture of

$$\frac{dy}{dx} = 1 + y^4.$$

The following important features can be deduced from the ordinary differential equation :

i) $\dfrac{dy}{dx} \geq 1 > 0$ hence y is strictly monotone increasing

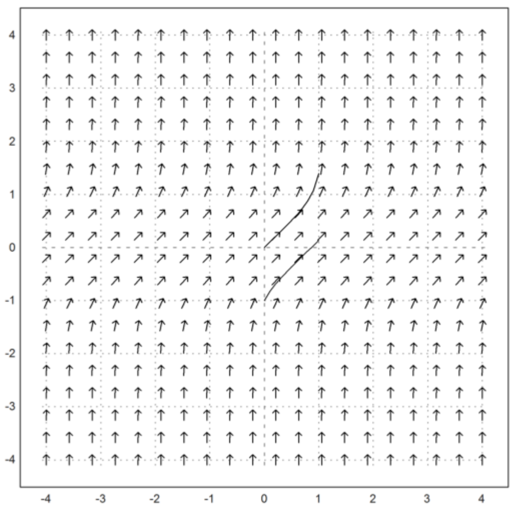
ii) $\dfrac{dy}{dx} = 1$ at $(0, 0)$,

iii) $\dfrac{dy}{dx}$ does not depend on variable x,

iv) $\dfrac{d^2y}{dx^2} = 4y^3 < 0$ if $y < 0$ implying that y is concave for negative y,

v) $\dfrac{d^2y}{dx^2} = 4y^3 > 0$ if $y > 0$ implying that y is convex for positive y.

Direction field and two sections of solution curves:



32. Suppose that two binary operations, denoted by $\bigoplus$ and $\bigotimes$, are defined on a nonempty set S, and that the following conditions are satisfied for all x, y and z in S:

i) $x \bigoplus y$ and $x \bigotimes y$ are in S,

ii) $x \bigoplus (y \bigoplus z) = (x \bigoplus y) \bigoplus z$ and $x \bigotimes (y \bigotimes z) = (x \bigotimes y) \bigotimes z$,

iii) $x \bigotimes y = y \bigotimes x$.

Further, *iteration* is defined on $\bigoplus$ as well as $\bigotimes$.

iv) $kx \Rightarrow (k+1)x = kx \bigoplus x$.

v) $x^k \Rightarrow x^{k+1} = x^k \bigoplus x$.

Then $n(x \bigoplus y) = nx \bigoplus ny$ and $x^m \bigotimes x^n = x^{m+n}$ are TRUE but $(x \bigotimes y)^n = x^n \bigotimes x^m$ is FALSE.

Discussion: This is a good example for heavy abstract algebraic machinery amounting to nothing, well, next to nothing. The "additive" structure defined by $\bigoplus$ is associative and commutative, it is not a group, for it does not have 0 or inverses. The "multiplicative" structure defined by $\bigotimes$, is even less, it is only associative, no 1 nor inverses. No distributive law to couple the two structures. The TRUE statements follow by induction on n, m. The FALSE statement can be disproved by

$$(x \bigotimes y)^2 = (x \bigotimes y) \bigotimes (x \bigotimes y) \neq (x \bigotimes x) \bigotimes (y \bigotimes y).$$

This exercise is designed to confuse the student.

33. Compute the greatest common divisor (g.c.d) also known as the highest common factor (H.C.F) of $a = 273$ and $b = 110$.

$$273 = 2 * 110 + 53$$

Remainder $53 < 110$.

$$110 = 2 * 53 + 4$$

Remainder $4 < 53$.

$$53 = 13 * 4 + 1$$

Remainder $1 < 4$.

$$4 = 4 * 1 + 0$$

Remainder $0 < 1$. g.c.d is the last non-zero remainder , 1

$$(273, 110) = 1.$$

The sequnce of computed values of remainder r is

$$53,\ 4,\ 1,\ 0.$$

$$273 = 3 * 7 * 13,\ 110 = 2 * 5 * 11$$