Pinter Consulting
New Series No. 21.
Spartan Old School
Seminars and Tutorials

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April 22, 2020

Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

- **"Der springt noch auf!"**
- Private studies for professional development:
- Socratic Programme
  - Analysis
  - Algebra and Number Theory
  - Geometry
  - Differential and Integral Equations
- Continuos improvement, corrections and last revision April 22, 2020.

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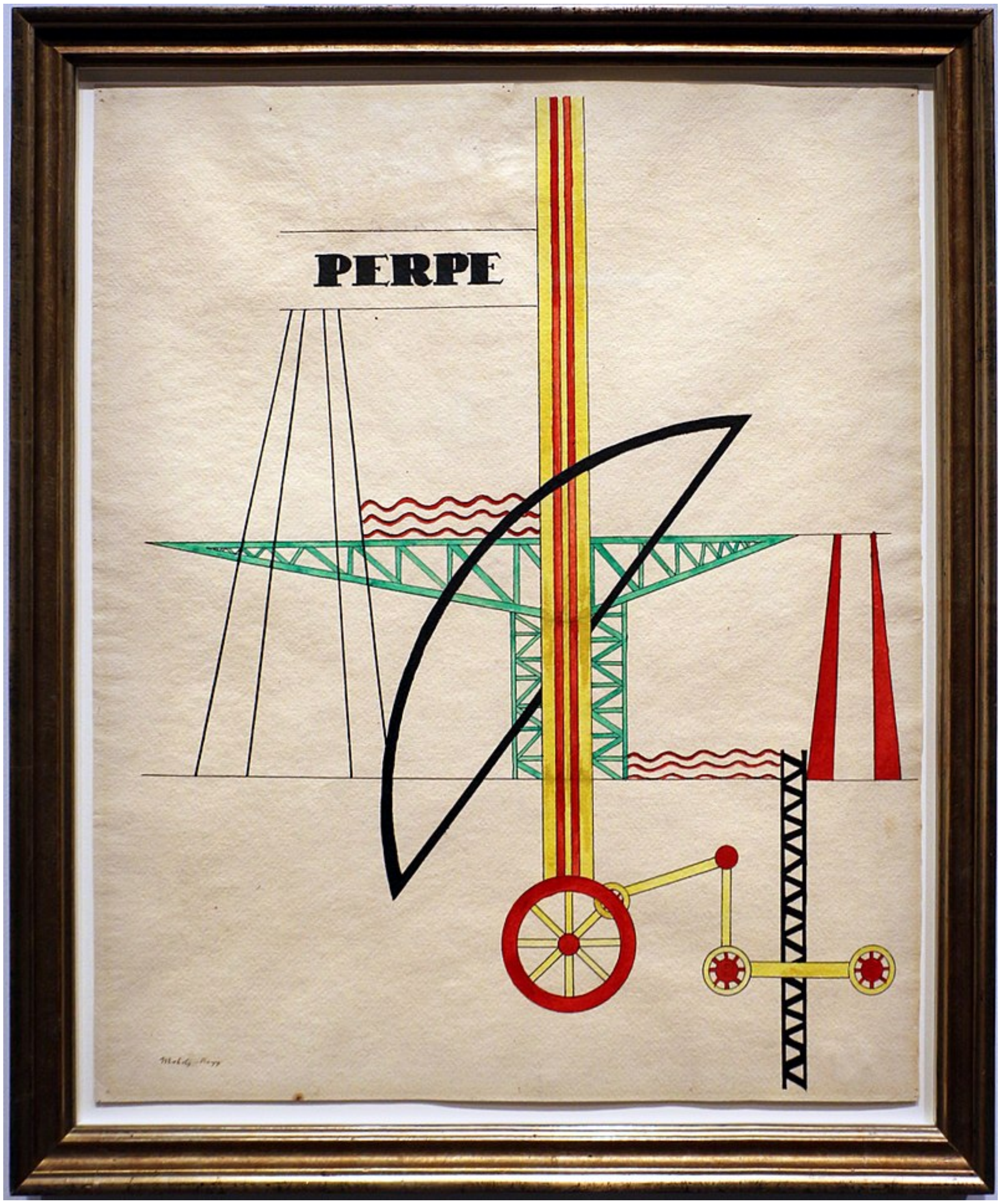
Introduction

Pinter Consulting of Calgary, Alberta practices Mathematics, promotes clear thinking and offers Consultations, Tutorials and Seminars in Mathematics.



Moholy-Nagy: Perpetuum Mobile

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20.0 Assignment 51.

- Problem-solving Seminar
- *Birkhoff-Rota: Ordinary Differential Equations*
- Last revision April 22, 2020

Exercises B

Exercise 1. *Find the solution of the DE $xy' + 3y = 0$ that satisfies the initial condition $f(1) = 1$*

$$xy' + 3y = 0$$

$$xy' = -3y$$

$$\int \frac{y'}{y} dy = -\int \frac{3}{x} dx$$

$$ln(y) = -3ln(x) + c_1$$

$$y = \frac{c_1}{x^3}.$$

$$f(1) = 1 \;\Rightarrow\; 1 = \frac{c_1}{1^3} \Rightarrow c_1 = 1.$$

$$y = \frac{1}{x^3}.\checkmark$$

Exercise 2. *Find equations describing all solutions of $y' = (x + y)^2$.*
Substitute $u = x + y$, u is a function of x, $u = u(x)$.

$$y = u - x$$

$$y' = u' - 1 = u^2$$

Recall

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\tan(x)' = \left[\frac{\sin(x)}{\cos(x)}\right]' = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos(x)\cos(x)} = 1 + \tan^2(x).$$

Therefore

$$u = \tan(x + c_1)$$

$$y = \tan(x + c_1) - x.$$

Exercise 3. *Find all solutions of the DE $xy' + (1 - x)y = 0$.*

$$(a)\ xy' + (1 - x)y = 0$$

$$\frac{y'}{y} = -\frac{1 - x}{x}$$

1-st order separable ODE, homogeneous, upon integrating both sides

$$\int \frac{y'}{y} = -\int \frac{1 - x}{x}$$

$$\ln(y) = -\ln(x) + x + c_1$$

$$y = \frac{\exp(x + c_1)}{x}.$$

Check:

$$y' = \frac{\exp(x + c_1)(x - 1)}{x^2}$$

$$xy' = \frac{\exp(x + c_1)(x - 1)}{x}$$

$$\frac{\exp(x + c_1)(x - 1)}{x} + (1 - x)\frac{\exp(x + c_1)}{x} = 0.\surd$$

$$(b)\ xy' + (1 - x)y = 1$$

this is a 1-st order separable ODE, inhomogeneous equation. Note that

$$y = \frac{\exp(x + c_1)}{x}.$$

solves homogeneous equation. Next, we present two methods for the solution of the inhomogeneous problem.

The problem to be solved is

$$y' + g(x)y = h(x);\; h(x) \neq 0.$$

Method I. Variation of contants: Assume

$$y = y_0 + Y$$

where y is the solution of the inhom. problem, Y is the known general solution of the homogeneous problem,

$$Y = C \exp\left(-\int g(c)dx\right)$$

y_0 is a particular solution of the inhom. problem. Write

$$y_0 = C(x) \exp\left(-\int g(c)dx\right)$$

Notice that $C(x)$ is variable now . Substitue y_0 into inhom. equation:

$$C'(x) \exp\left(-\int g(x)dx\right) - C(x)g(x) \exp\left(-\int g(x)dx\right) + C(x)g(x) \exp\left(-\int g(x)dx\right) = h(x)$$

$$C'(x) \exp\left(-\int g(x)dx\right) = h(x)$$

$$\frac{dC(x)}{dx} = h(x) \exp\left(\int g(x)dx\right)$$

$$C(x) = \int h(x) \exp\left(\int g(x)dx\right) dx$$

We just need one particular solution so we can set the constant of integration to zero. Thus

$$y_0 = \int h(x) \exp\left(\int g(x)dx\right) dx * \exp\left(-\int g(x)dx\right).$$

Method II. Undetermined functions : Write

$$y = uv$$

where $u = u(x),\ v = v(x)$ are undetermined functions.

$$y = uv' + u'v,$$

$$uv' + u'v + g(x)uv = h(x)$$

$$u[v' + g(x)v] + [vu' - h(x)] = 0$$

This equation holds (trivially) if

$$v' + g(x)v = 0$$

and

$$vu' - h(x) = 0.$$

All we have to do now is to solve these two equations one after the other.

$$v = \exp\left(-\int g(x)dx\right)$$

$$u = C + \int h(x)\exp\left(\int g(x)dx\right)dx$$

Formula.

$$y = \left[C + \int h(x)\exp\left(\int g(x)dx\right)\right]\exp\left(-\int g(x)dx\right)$$

solution by 2nd Method:

$$v(x) = \exp\left(-\int (x^{-1} - 1)dx\right) = \exp\left(-\ln(x) + x\right) = \frac{\exp(x)}{x}.$$

$$u' = \frac{h(x)}{v(x)} = \exp(-x).$$

$$u = -\exp(-x) + C$$

$$y = uv = (-\exp(-x) + C) * \frac{\exp(x)}{x} = \frac{C\exp(x) - 1}{x} = \frac{C - \exp(-x)}{x\exp(-x)}.$$

Check:

$$y' = \frac{C\exp(x)x - C\exp(x) + 1}{x^2}$$

$$x\frac{C\exp(x)x - C\exp(x) + 1}{x^2} + (1 - x)\frac{C\exp(x) - 1}{x} = 1$$

$$C\exp(x) - \frac{C\exp(x) - 1}{x} + \frac{C\exp(x) - 1}{x} - C\exp(x) + 1 = 1$$

$$1 = 1.\ \surd$$

Exercise 4. *(a) Solve the DEs of Exercise 3. for initial condition $y(1) = 1$, $y(1) = 2$.*

$$y(1) = -\frac{1}{1} + \frac{C\exp(1)}{1} = 1$$

$$C = 2\exp(-1).$$

another initial condition

$$y(1) = 2$$

$$-\frac{1}{2} + \frac{C\exp(2)}{2} = 2$$

$$\frac{C\exp(2)}{2} = \frac{5}{2}$$

$$C = \frac{5}{\exp(2)}.$$

(b) Do the same for $y(0) = 0$ and $y(0) = 1$, or prove that no solution exists. The function

$$y = \frac{1 - C\exp(x)}{x}$$

is not defined for $x = 0$, there is no solution on the y axis. We can calculate limits, or examine the qualitative behavior of y as x goes to zero.

$$y = \frac{1 - C\exp(x)}{x} \approx \frac{1 - C\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\dots\right)}{x} =$$

$$\frac{1-C}{x} + \frac{C\left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\dots\right)}{x} =$$

$$\frac{1-C}{x} + C\left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!}\dots\right).$$

There are two cases, if $C = 1$,

$$y = \left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!}\dots\right) \to 1 \text{ as } x \to 0.$$

If $C \neq 1$ then

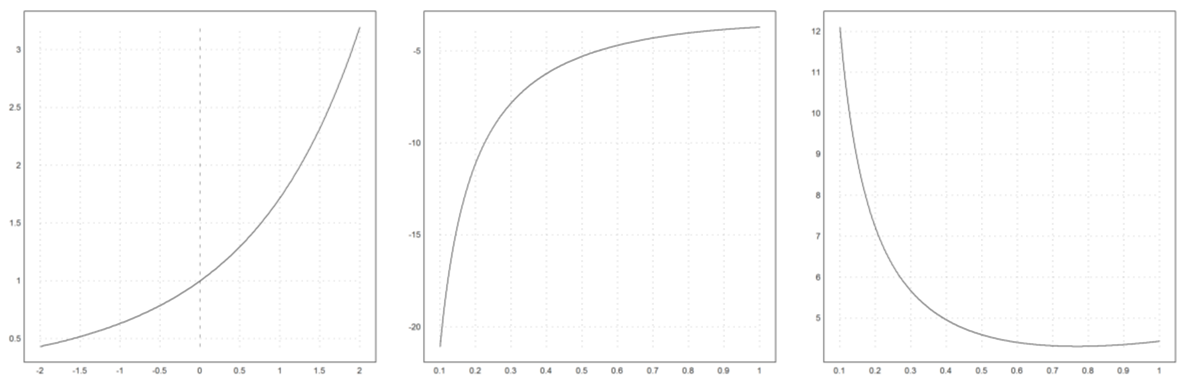
$$\left|\frac{1-C}{x}\right| \to \infty, \; C\left(\frac{1}{1!}+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}\dots\right) \to C \text{ as } x \to 0.$$

and

$$\left|C\left(\frac{1}{1!}+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}\dots\right)\right| \ll \left|\frac{1-C}{x}\right|$$

$$y \approx \frac{1-C}{x} \text{ as } x \to 0.$$

In conclusion, solutions with $y(0) = 0$, or $y(0) = 1$ do not exist.



Qualitative pictures, different scales, point $(0, 1)$ is a discontinuity. The left sketch is for $C = 1$, the middle one is $C = -1$, the right sketch is for $C = 2$.

Certain indefinite integral.

$$\int exp(ax)\cos(bx)dx = \frac{b\sin(bx) + a\cos(bx)}{a^2+b^2}\exp(ax) + C_1$$

Partial integration for indefinite integrals:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.$$

To verify write

$$d(uv) = u'vdx + uv'dx$$

$$\int uv'dx = uv - \int u'vdx.$$

Choose u, v, and differentiate $u(x)$, integrate $v'(x)$.

$$\int exp(ax)\cos(bx)dx,\ u = exp(ax),\ v' = \cos(bx)$$

$$\int exp(ax)\cos(bx)dx = \frac{1}{b}exp(ax)\sin(bx) - \frac{a}{b}\int exp(ax)\sin(bx)dx$$

2nd partial integral, parallel process:

$$v' = exp(ax),\ u = \cos(bx)$$

$$\int exp(ax)\cos(bx)dx = \frac{1}{a}exp(ax)\cos(bx) + \frac{b}{a}\int exp(ax)\sin(bx)dx$$

Scale equations by $\left(\frac{b}{a}\right)$, respectively, and $\left(\frac{a}{b}\right)$ and sum them to cancel second terms.

$$\left(\frac{b}{a} + \frac{a}{b}\right)\int exp(ax)\cos(bx)dx = \frac{1}{a}exp(ax)\sin(bx) + \frac{1}{b}exp(ax)\cos(bx)$$

$$\int exp(ax)\cos(bx)dx = \left(\frac{b}{a} + \frac{a}{b}\right)^{-1}\left(\frac{1}{a}exp(ax)\sin(bx) + \frac{1}{b}exp(ax)\cos(bx)\right)$$

$$\left(\frac{b}{a} + \frac{a}{b}\right)^{-1} = \left(\frac{b^2}{ab} + \frac{a^2}{ab}\right)^{-1} = \left(\frac{a^2 + b^2}{ab}\right)^{-1} = \left(\frac{ab}{a^2 + b^2}\right)$$

$$\int exp(ax)\cos(bx)dx = \left(\frac{ab}{a^2 + b^2}\right)\left(\frac{1}{a}\sin(bx) + \frac{1}{b}\cos(bx)\right)exp(ax)$$

Similar argument leads to

$$\int exp(ax)\sin(bx)dx = \frac{a\sin(bx) - b\cos(bx)}{a^2 + b^2}\exp(ax) + C_1.$$

5. *(a) Find the general solution of the DE $y' + y = \sin(2t)$.*
Write

$$y = uv,\ y' = u'v + uv'.$$

$$u'v + uv' + uv = \sin(2t),$$
$$u(v' + v) + (u'v - \sin(2t)) = 0$$

then

$$(v' + v) = 0;\ (u'v - \sin(2t)) = 0$$

trivially implies equation.

$$v' + v = 0$$

$$v(t) = \exp(-t),\ v'(t) = -\exp(-t);\ C \equiv 1.$$

$$u' = \frac{\sin(2t)}{\exp(-t)} = \sin(2t)\exp(t)$$

$$\int \sin(2t)\exp(t)dt = \frac{a\sin(bx) - b\cos(bx)}{a^2 + b^2}\exp(ax) + C_1,\ b = 2,\ a = 1$$

$$\int \sin(2t)\exp(t)dt = \frac{\sin(2t) - 2\cos(2t)}{5}\exp(t) + C_1$$

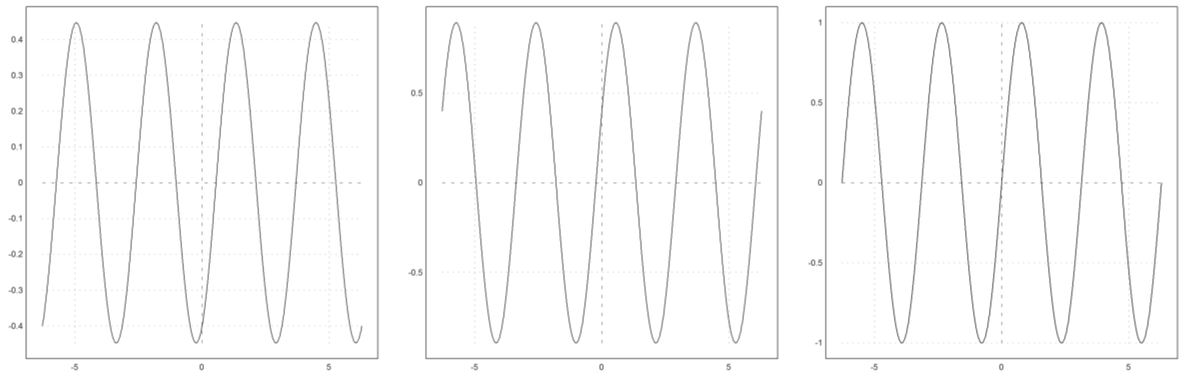
$$y(t) = \left(\frac{\sin(2t) - 2\cos(2t)}{5}\exp(t) + C_1\right)\exp(-t)$$

$$y(t) = \frac{\sin(2t) - 2\cos(2t)}{5} + C_1\exp(-t).$$

Check:

$$y' = \frac{4\sin(2t) + 2\cos(2t)}{5} - C_1\exp(-t)$$

$$y' + y = 5\frac{\sin(2t)}{5} = \sin(2t).\ \checkmark$$



$y(t), y'(t)$ and $y' + y = \sin(2t)$.

Exercise 5. *(a) Find particular and general solution of the DE* $y' = ay + b\sin(kt)$*;* a, b *are real constants and* $k \neq 0$. Write

$$y' - ay = b\sin(kt)$$

where $-a = g(t)$ and $b\sin(kt) = h(t)$.

$$C(t) = \int h(t)\exp\left(\int g(t)dt\right)dt$$

$$C(t) = \int b\sin(kt)\exp(-at)dt = b\int b\sin(kt)\exp(-at)dt$$

$$C(t) = \frac{b}{a^2+k^2}(-a\sin(kt) - k\cos(kt))$$

$$C'(t) = \frac{b}{a^2+k^2}(-ak\cos(kt) + k^2\sin(kt))$$

$$-aC(t) = \frac{b}{a^2+k^2}(a^2\sin(kt) + ak\cos(kt))$$

$$C'(t) - aC(t) = b\frac{a^2+k^2}{a^2+k^2}\sin(kt) = b\sin(kt)$$

Therefore $C(t)$ is a particular solution. General solution is

$$y(t) = \exp(ax)\left[\frac{b\exp(-ax)}{a^2+k^2}(-a\sin(kt) - k\cos(kt) + C_1)\right]$$

$$y(t) = \left[\frac{b}{a^2+k^2}(-a\sin(kt) - k\cos(kt) + C_1\exp(ax))\right]$$

Check:

$$y'(t) = \left[\frac{b}{a^2+k^2}(-ak\cos(kt) + k^2\sin(kt) + aC_1\exp(ax))\right]$$

$$-ay(t) = \left[\frac{b}{a^2+k^2}(a^2\sin(kt) + ak\cos(kt) + -aC_1\exp(ax))\right]$$

$$y'(t) - ay(t) = b\left[\frac{a^2+k^2}{a^2+k^2}\right]\sin(kt) = b\sin(kt).\ \checkmark$$

Exercise 6. *Find a polynomial solution of the DE*

$$y' + 2y = x^2 + 4x + 7.$$

$$y(x) = ax^2 + bx + c,$$

because the right hand side indicates that y is a quadratic polynomial.

$$y'(x) = 2ax + b,$$

$$y' + 2y = 2ax + b + 2(ax^2 + bx + c) = 2ax^2 + (2a + 2b)x + 2c + b$$

$$2a = 1$$

$$2a + 2b = 2(a + b) = 4$$

$$2c + b = 7$$

$$a = \frac{1}{2};\ b = \frac{3}{2},\ c = \frac{11}{4}.$$

Check:

$$y = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{11}{4}$$

$$y' = x + \frac{3}{2}$$

$$y' + 2y = x + \frac{3}{2} + 2\left(\frac{1}{2}x^2 + \frac{3}{2}x + \frac{11}{4}\right) = x^2 + 4x + 7.\ \checkmark$$

Exercise 7. *Show that if k is a nonzero constant and $q(x)$ a polynomial of degree n, then the DE $xy' + y = q(x)$ has exactly one polynomial solution of degree n.*

Not clear.

Write

$$k = b_0$$

$$q(x) = \sum_{k=1}^{n} b_k x^k + b_0$$

$$y(x) = \sum_{k=1}^{n} a_k x^k + a_0$$

$$y'(x) = \sum_{k=1}^{n} k a_k x^{k-1}$$

$$xy'(x) = \sum_{k=1}^{n} k a_k x^k$$

$$(k+1)a_k = b_k,\ k = 1 \dots n$$

$$a_0 = b_0.$$

Exercise 8.

$$\frac{dr}{d\theta} = r^2 \sin\left(\frac{1}{r}\right), \quad \text{polar coordinates}$$

$$u = \frac{1}{r}$$

$$r = \frac{1}{u}$$

$$dr = -\frac{du}{u^2}$$

$$-\frac{1}{u^2}\frac{du}{d\theta} = \frac{1}{u^2}\sin(u)$$

$$-\frac{du}{\sin(u)} = d\theta$$

$$\sin(u) = \sin\left(2\frac{u}{2}\right) = 2\sin\left(\frac{u}{2}\right)\cos\left(\frac{u}{2}\right)$$

$$\frac{1}{\sin(u)} = \frac{1}{2\sin\left(\frac{u}{2}\right)\cos\left(\frac{u}{2}\right)} * \frac{\frac{1}{\cos^2\left(\frac{u}{2}\right)}}{\frac{1}{\cos^2\left(\frac{u}{2}\right)}} = \frac{\tan'\left(\frac{u}{2}\right)}{\tan\left(\frac{u}{2}\right)}$$

$$-\frac{du}{\sin(u)} = -\frac{\tan'\left(\frac{u}{2}\right)}{\tan\left(\frac{u}{2}\right)}du = d\theta$$

$$-\int \frac{du}{\sin(u)} = -\int \frac{\tan'\left(\frac{u}{2}\right)}{\tan\left(\frac{u}{2}\right)}du = -\log\left(\tan\left(\frac{u}{2}\right)\right) = \theta + C_1$$

$$\frac{1}{r} \ \rightarrow \ u$$

$$-\log\left(\tan\left(\frac{1}{2r}\right)\right) = \theta + C_1, \text{ implicit solution.}$$

$$r(\theta) = [2 * \arctan(\exp(-\theta))]^{-1}, \text{ explicit solution.}$$

Calculate limit:

$$\lim_{r \to \infty} \frac{r^2 \sin\left(\frac{1}{r}\right)}{r} = \lim_{r \to \infty} \frac{\sin\left(\frac{1}{r}\right)}{\frac{1}{r}} = 1$$

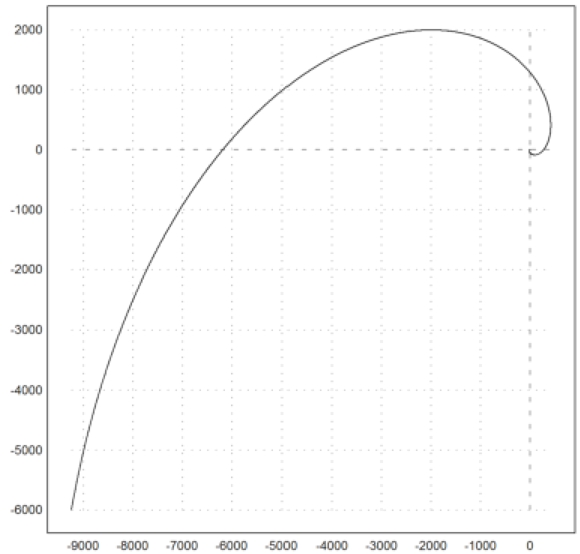
Asymptotically,

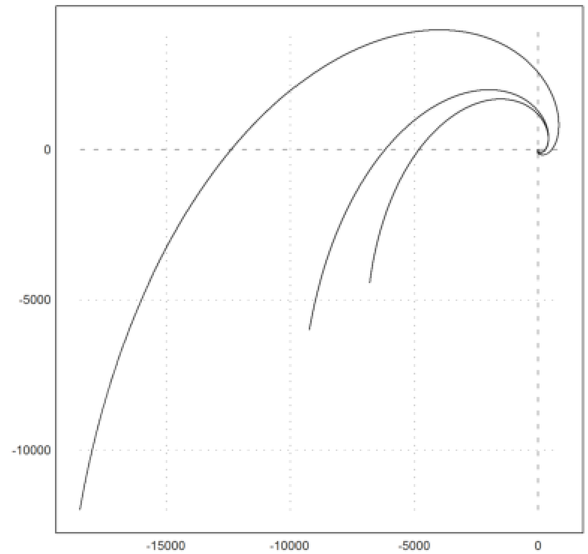
$$r^2 \sin\left(\frac{1}{r}\right) \approx r; \ r \to \infty.$$

Solve

$$\frac{dr}{d\theta} = r$$

$$r = \exp(\theta), \text{ logarithmic spiral.}$$





The diagram on the left displays the solution curve, $r(\theta)$, the diagram on the right shows three spirals (Note different scales!). For reference, the outer one is the standard *logarithmic spiral* $r(\theta) = \exp(\theta)$; the inner spiral is $\exp(0.9 * \theta)$. The solution curve, is in the middle. enveloped by the two logarithmic spirals. In conclusion, we claim, that *quantitatively* , the solution to Ex.8. is a logarithmic spiral.

Snippet Euler Math Toolbox:

```
% definitions:
>function f(x):= 1.0/(2*arctan(exp(-x)));
>function h(x):= exp(0.9*x);
>function g(x):= exp(x);
  % displays:
>plot2d("g(x)*cos(x)","g(x)*sin(x)", a=0, b=10)
>plot2d("f(x)*cos(x)","f(x)*sin(x)", a=0, b=10, >add)
>plot2d("h(x)*cos(x)","h(x)*sin(x)", a=0, b=10, >add)
```

Exercise 9.

$$\frac{dr}{d\theta} = \frac{2}{\log(r)}, \quad \text{polar coordinates}$$

$$\log(r)dr = 2d\theta$$

Recall partial integration formula

$$d(uv) = u'vdx + uv'dx$$

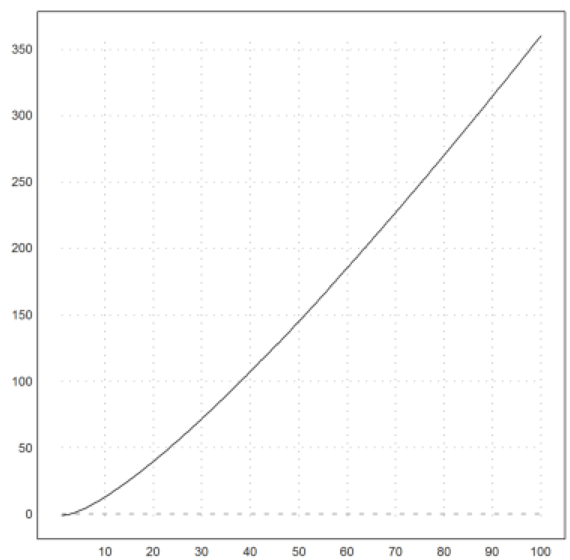
$$\int uv'dx = uv - \int u'vdx.$$

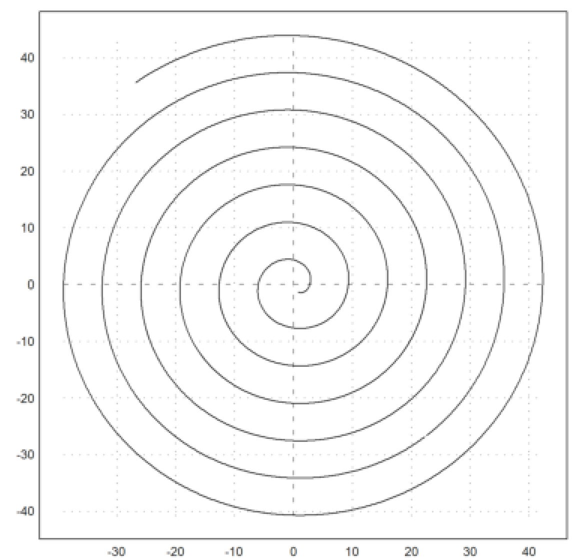
$$u = \log(r), v' = 1$$

$$\int \log(r)dr = \int 1 * \log(r)dr = r\log(r) - \int \frac{r}{r}dr = r\log(r) - r.$$

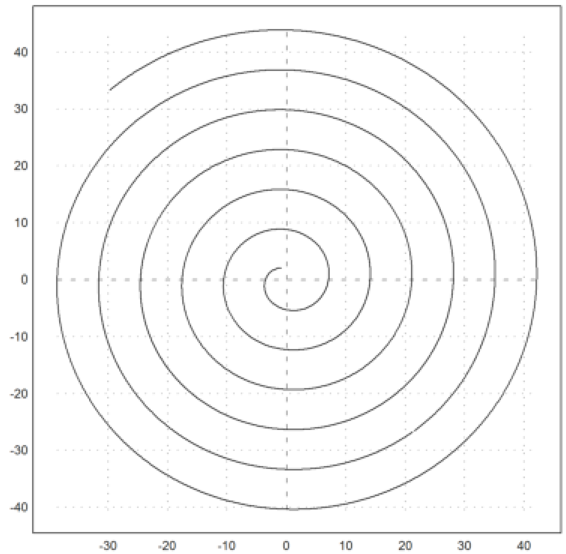
$$r(\log(r) - 1) = \theta \quad \text{implicit solution.}$$

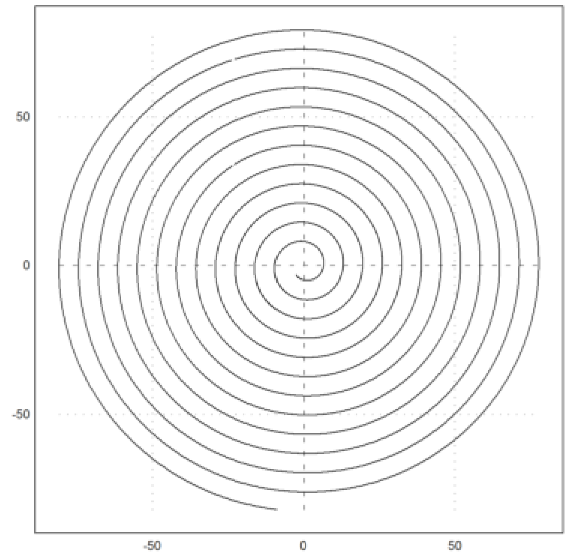
The function $\theta(r)$ is strictly monotone increasing, its inverse exists. The solution, a spiral, is displayed on the right.





For comparison, $r = 2\theta$ and $r = 4\theta$ are depicted next.





Snippet Euler Math Toolbox:

```
% relation
plot2d( "x*(log(x)-1)", a=1, b=100)
% spiral
>function f(x):= x*(log(x)-1)
>plot2d( "sqrt(x*x+f(x)*f(x))*cos(f(x))",

 "sqrt(x*x+f(x)*f(x))*sin(f(x))",a=1, b=20)
```

By visual inspection, we claim that the solution spiral to Ex.9. is resembling an *Archimedean spiral* .

Exercise 10. *Show that the ellipses* $5x^2+6xy+5y^2 = C$ *are integral curves of the DE*

$$(5x + 3y) + (3x + 5y)y' = 0$$

Differentiate

$$2 * 5y + 6y + 6xy' + 2 * 5yy' = 0$$

$$2(5x + 3y) + 2(3x + 5y)y' = 0.$$

20.1 Assignment 43.

Summary

- Pólya - Szegő: *Aufgaben und ...*
- Problems on the Integral Parts of Numbers (Part VIII)
- Last revision April 22, 2020

Definitions

Let x be a number. Denote by $[x]$ the *integral part* of x that is the integer that satisfies the inequality

$$x - 1 < [x] \leq x < [x] + 1$$

Examples:

$$[\pi] = 3, \ [2] = 2, \ [-0.53] = -1$$

1.

Let n be an integer and x arbitrary. We then have

$$[x + n] = [x] + n.$$

Proof:

$$x + n - 1 < [x + n] \leq x + n < [x + n] + 1.$$

$$x - 1 < [x + n] - n \leq x.$$

By definition,

$$x - 1 < [x] \leq x.$$

Between $x-1$ and x there is one and only one integer, $[x]$. But $[x+n]-n$ is also an integer between $x - 1$ and x. Therefore the two should be the same

$$[x + n] - n = [x]$$

hence

$$[x + n] = [x] + n. \checkmark$$

2.

In the expansion of the determinant of $n-th$ *order, the product of the elements in the secondary diagonal has the sign* $(-1)^{[\frac{n}{2}]}$.

Proof:

$$\det A = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} = \sum (-1)^{t(j)} a_{1,j_1} a_{2,j_2} \dots a_{n,j_n}$$

where $t(j)$ is the number of inversions in the permutation $t(j) = (j_1, j_2, \dots, j_n)$. The secondary diagonal is

$$a_{1,n} a_{2,n-1} \dots a_{n,1}$$

and

$$t(j) = (n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$$

by the well-known summation formula. Thus we are required to show that

$$\frac{n(n-1)}{2} \equiv \left[\frac{n}{2}\right], \text{ (modulo 2)}.$$

Next, we consider four cases, $n = 0, 1, 2, 3$.

$$n = 0; \ \frac{0(0-1)}{2} \equiv \left[\frac{0}{2}\right] \equiv 0 \text{ (modulo 2)}.$$

$$n = 1; \ \frac{1(1-1)}{2} \equiv \left[\frac{1}{2}\right] \equiv 0 \text{ (modulo 2)}.$$

$$n = 2; \ \frac{2(2-1)}{2} \equiv \left[\frac{2}{2}\right] \equiv 1 \text{ (modulo 2)}.$$

$$n = 3; \ \frac{3(3-1)}{2} \equiv \left[\frac{3}{2}\right] \equiv 1 \text{ (modulo 2)}.$$

The required congruence holds for $n = 0, 1, 2, 3$.

Take $n + 4$.

$$\frac{(n+4)(n+3)}{2} - \frac{n(n-1)}{2} = \frac{8n+12}{2} = 4n + 6 \equiv 0 \text{ (modulo 2)},$$

the difference is an even number. Therefore

$$\frac{(n+4)(n+3)}{2} \equiv \frac{n(n-1)}{2} \equiv \left[\frac{n}{2}\right] \text{ (modulo 2)}$$

and the sign of the product is $(-1)^{[\frac{n}{2}]}$ for every n. $\surd$

3.

We have

$$[2x] - 2[x] = 0 \textit{ or } 1$$

according as

$$x - [x] < \frac{1}{2} \textit{ or } \geq \frac{1}{2}.$$

Proof:

Recall that by definition

$$x - 1 < [x] \leq x < [x] + 1$$

$[x]$ is the greatest integer not exceeding x, which can also be termed *left-adjacent* to x.

Case i)

$$0 \leq x - [x] < \frac{1}{2}$$

$$0 \leq 2x - 2[x] < 1$$

The greatest integer not exceeding $2x - 2[x]$ cannot be 1 because 1 exceeds $2x - 2[x]$. (Check equation above.) So the required integer integer is less than 1; 0 is less than 1, furthermore 0 less than or equal to $2x - 2[x]$. Can the greatest integer not exceeding $2x - 2[x]$ be other than 0 ? No, because there is no integer between 0 and 1. Therefore

$$[2x - 2[x]] = 0$$

and by 1.

$$[2x - 2[x]] = [2x] - 2[x] = 0.$$

Case ii)

$$x - [x] \geq \frac{1}{2}$$

$$2 > 2x - 2[x] \geq 1$$

$$[2 > 2x - 2[x]] = 1$$

By 1. again

$$[2 > 2x - 2[x]] = [2x] - 2[x] = 1. \surd$$

4.

If $0 < \alpha < 1$, *then we have*

$$[x] - [x - \alpha] = 0 \ \textit{or} \ 1$$

according as

$$x - [x] \geq \alpha \ \textit{or} \ < \alpha.$$

Proof:

Write

$$x - [x] = \{x\}$$

where $\{x\}$ is the *fractional part* of x. Clearly

$$x = [x] + \{x\}$$

and $\{x\}$ is non-negative.

$$[x] - [x - \alpha] = [x] - [[x] + \{x\} - \alpha].$$

Apply 1 to $[[x] + (\{x\} - \alpha)]$, $[x]$ is integer, $(\{x\} - \alpha)$ is arbitrary. If

$$0 \leq \{x\} - \alpha < \{x\} < 1$$

then

$$[x] - [x - \alpha] = 0.$$

If

$$-1 < \{x\} - \alpha < 0$$

then

$$[[x] + \{x\} - \alpha] = [x] - 1,$$

and

$$[x] - [x - \alpha] = 1. \checkmark$$

5.

Let x be a number that does not lie at the midpoint between two consecutive integers. Express the integer nearest to x in terms of the symbol [ ].

Write

$$\beta = x - [x];\ 0 \leq \beta < 1,$$

where β is the fractional part of x. Observe that

$$0 \leq 2\beta < 2.$$

Therefore $[2\beta] = 1$ if $\beta > \frac{1}{2}$, and $[2\beta] = 0$ if $\beta < \frac{1}{2}$.

In the first case, $[x] + 1$ is the closest integer to x,

$$[x]+1 = [x]+[2\beta] = [x]+[2(x-[x])] = [x]+[2x-2[x]] = [x]+[2x]-2[x].$$

In the second case $[x]$ is the closest integer to x,

$$[x] = [x] + [2\beta] = [x] + [2(x - [x])] = [x] + [2x - 2[x]] = [x] + [2x] - 2[x].$$

In both cases

$$[x] + [2x] - 2[x] = [2x] - [x]$$

Therefore the integer nearest to x in terms of of the symbol [ ] is $[2x] - [x]$.

6.

One may term $[x]$ *the integer left-adjacent* to x. Express the integer *right-adjacent* to x in terms of [ ]. (See for example postal rates or interurban telephone rates.)

The required number n satisfies

$$n - 1 < x \le n$$

. Hence - by reversing the inequality -

$$-n \le -x < -n + 1,$$

therefore

$$n = -[-x].$$

7.

Prove that

$$i) \;\; [\alpha] + [\beta] = \textit{either } [\alpha + \beta] \textit{ or } [\alpha + \beta] - 1$$

and

$$ii) \;\; [\alpha] - [\beta] = \textit{either } [\alpha - \beta] \textit{ or } [\alpha - \beta] + 1.$$

First, we claim that $[\alpha + \beta]$ and $[\alpha] + [\beta]$ are integers not exceeding $(\alpha + \beta)$. By definition,

$$[\alpha] \le \alpha < [\alpha] + 1,$$

$$[\beta] \le \beta < [\beta] + 1,$$

and

$$[\alpha + \beta] \le \alpha + \beta < [\alpha + \beta] \; + 1$$

for $\alpha, \; \beta$ real. Hence

$$[\alpha + \beta] \le \alpha + \beta$$

$$[\alpha] + [\beta] \le \alpha + \beta.$$

Further,

$$\alpha + \beta \geq [\alpha + \beta] > [\alpha + \beta] - 1,$$

the last two integers not exceeding $\alpha + \beta$. The difference between $\alpha + \beta$ and $[\alpha] + [\beta]$ is less than 2:

$$0 \leq (\alpha + \beta) - ([\alpha] + [\beta]) = (\alpha - [\alpha]) + (\beta - [\beta]) < 1 + 1 < 2.$$

Therefore $[\alpha] + [\beta]$ is either $[\alpha + \beta]$ or $[\alpha + \beta] - 1$. This proves *i*).

As for *ii*), we take a different approach because we cannot subtract inequalities. Write

$$\alpha = [\alpha] + (\alpha - [\alpha])$$

where $0 \leq (\alpha - [\alpha]) < 1$, the fractional part. Similarly,

$$\beta = [\beta] + (\beta - [\beta]).$$

Then

$$\alpha - \beta = [\alpha] - [\beta] + (\alpha - [\alpha]) - (\beta - [\beta])$$

$$[\alpha - \beta] = [[\alpha] - [\beta] + (\alpha - [\alpha]) - (\beta - [\beta])]$$

$$[\alpha - \beta] = [\alpha] - [\beta] + [(\alpha - [\alpha]) - (\beta - [\beta])]$$

by **1.** Since

$$-1 < (\alpha - [\alpha]) - (\beta - [\beta]) < 1$$

$$[(\alpha - [\alpha]) - (\beta - [\beta])] = -1 \text{ or } 0,$$

Therefore

$$[\alpha - \beta] = [\alpha] - [\beta] - 1$$

or

$$[\alpha - \beta] = [\alpha] - [\beta]$$

and *ii*) is proven.

8.

We have

$$[2\alpha] + [2\beta] \geq [\alpha] + [\alpha + \beta] + [\beta]$$

Proof:

First, we show that the above equation is *translation invariant* . Recall that

$$[x + n] = [x] + n$$

by **Problem 1.**

Write

$$\alpha \to \alpha + n;\ \beta \to \beta + m;$$

Then left-hand side

$$[2(\alpha + n)] + [2(\beta + m)] = [2\alpha] + 2n + [2\beta] + 2m = [2\alpha] + [2\beta] + 2n + 2m$$

and right-hand side

$$[\alpha + n] + [\alpha + n + \beta + m] = [\alpha] + [\alpha + \beta] + [\beta] + 2n + 2m$$

change by the same amount. Therefore it is enough to consider the theorem for

$$0 \leq \alpha < 1,\ 0 \leq \beta < 1.$$

Problem 8. reduces to

$$[2\alpha] + [2\beta] \geq [\alpha + \beta].$$

Clearly,

$$[\alpha + \beta] = 0 \text{ or } 1$$

If $[\alpha + \beta] = 0$, there is nothing to prove.

If $[\alpha + \beta] = 1$ then $\alpha + \beta \geq 1$ and at least one of α and β is greater than or equal to $\frac{1}{2}$, - say -

$$\alpha \geq \frac{1}{2}.$$

$$[2\alpha] + [2\beta] \geq [2\alpha] \geq 1.$$

This proves the reduced statement:

$$[2\alpha] + [2\beta] \geq [\alpha + \beta];\ 0 \leq \alpha < 1,\ 0 \leq \beta < 1.$$

By *translation invariance* , the theorem follows.

9.

Let n be a positive integer, x arbitrary, then we have

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx].$$

Proof I:

By definition $[nx]$ is an integer, thus

$$[nx] = qn + r; \ 0 \le r \le n - 1$$

where q and r are integers, $0 \le r \le n-1$. (n divisor, q quotient, r remainder; negative q allowed) Write $\theta = \{nx\}; \ 0 \le \theta < 1$, for fractional part. Then

$$nx = [nx] + nx = qn + r + \theta$$

$$x = q + \frac{r}{n} + \frac{\theta}{n}.$$

Let

$$i = 0, 1, \ldots n - r - 1.$$

$$x + \frac{i}{n} = (q + \frac{r}{n} + \frac{\theta}{n}) + \frac{i}{n} = q + \frac{r + \theta + i}{n} = q + \frac{(r + i) + \theta}{n}$$

Since

$$r + i \le n - 1, \text{ and } \theta < 1$$

$$x + \frac{i}{n} \le q + \frac{n - 1 + \theta}{n} < q + \frac{n - 1 + 1}{n} = q + 1.$$

So on one hand

$$x + \frac{i}{n} < q + 1$$

and on the other hand

$$x + \frac{i}{n} \ge x \ge q.$$

$$q \le x + \frac{i}{n} < q + 1$$

Therefore

$$q = \left[x + \frac{i}{n}\right]$$

for the first $n - r$ terms.

Next, let

$$i = n - r, n - r + 1, \ldots n - 1.$$

$$x + \frac{1}{n} = q + \frac{r}{n} + \frac{\theta}{n} + \frac{i}{n} = q + \frac{(r+i)+\theta}{n} \ge q + \frac{n+0}{n} \ge q + 1.$$

Moreover,

$$x + \frac{1}{n} = x + \frac{i}{n} = q + \frac{r}{n} + \frac{\theta}{n} + \frac{i}{n} < q + \frac{n-1}{n} + \frac{1}{n} + \frac{n-1}{n} = q + \frac{2n-1}{n},$$

and

$$x + \frac{i}{n} < q + 2.$$

Therefore

$$q + 1 \le x + \frac{i}{n} < q + 2.$$

$$\left[x + \frac{i}{n}\right] = q + 1.$$

The first $(n - r)$ terms contribute $(n - r)(q)$,

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-r}{n}\right] = (n - r)(q)$$

the remaining r terms $r(q + 1)$.

$$\left[x + \frac{(n-r+1}{n}\right] + \left[x + \frac{n-r+2}{n}\right] + \cdots + \left[x + \frac{n}{n}\right] = r(q + 1)$$

$$(n - r)(q) + r(q + 1) = nq - rq + rq + r = nq + r = [xn]$$

and the theorem is proven.

Proof II:

We may assume $0 \leq x < 1$, see **Problems 8., 9.**. Consider the following increasing sequence:

$$[x];\ \left[x+\frac{1}{n}\right];\ \left[x+\frac{2}{n}\right];\ \ldots \left[x+\frac{k-1}{n}\right];\ 1; \left[x+\frac{k}{n}\right];\ldots \left[x+\frac{n-1}{n}\right]$$

We are interested in the value of k such that

$$\left[x+\frac{k-1}{n}\right] < 1 \leq \left[x+\frac{k}{n}\right].$$

For such k,

$$[x] = \left[x+\frac{1}{n}\right] = \left[x+\frac{2}{n}\right] = \cdots = \left[x+\frac{k-1}{n}\right] = 0$$

and

$$\left[x+\frac{k}{n}\right] = \cdots = \left[x+\frac{n-1}{n}\right] = 1.$$

From

$$1 \leq \left[x+\frac{k}{n}\right]$$

we can deduce that

$$1 = \left[x+\frac{k}{n}\right],$$

because

$$0 < \left[x+\frac{k}{n}\right] < 2,$$

a positive integer less than 2.

$$n \leq n\left[x+\frac{k}{n}\right] \leq nx+k$$

$$-k \leq nx - n$$
$$-k = [nx - n].$$

From **Problem 1.**

$$-k = [nx - n] = [nx] - n$$

and finally,

$$n - k = [nx]$$

for the right hand side. For the left hand side we have

$$\left[x + \frac{k}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = 1 + 1 + \cdots + 1 = n - k.$$

Therefore the theorem is proven.

10.

Let n be a positive integer, x arbitrary. We have

$$\left[\frac{[nx]}{n}\right] = [x].$$

Proof:

First, we show *translation invariance* , m is an integer.

$$[n(x+m)] = [nx + nm] = [nx] + nm$$
$$\frac{[n(x+m)]}{n} = \frac{[nx] + nm}{n} = \frac{[nx]}{n} + m$$
$$\left[\frac{[nx]}{n} + m\right] = \left[\frac{[nx]}{n}\right] + m$$

for the left-hand side and

$$[x + m] = [x] + m$$

for the right-hand side. Thus

$$\left[\frac{[nx]}{n}\right] + m = [x] + m$$

$$\left[\frac{[nx]}{n}\right] = [x],$$

and we may assume

$$0 \leq x < 1$$

see **Problems 8., 9.**The right-hand side is

$$[x] = 0.$$

Furthermore

$$[nx] \leq nx; \ \frac{[nx]}{n} \leq x < 1; \ \left[\frac{[nx]}{n}\right] = 0.$$

The two sides are equal to zero. $\surd$

11.

Let m be a positive integer. The highest power that is a divisor of

$$[(1 + \sqrt{3})^{2m+1}]$$

is 2^{m+1}.

Preliminaries:

Lemma on Binomial Theorem

Definition and notation of binomial coefficient: The k-combination of n things is a selection of k of them without regard to order, $k! = 1 * 2 * \cdots * (k-1) * k$, is k-factorial, further $0! = 1$,$1! = 1$ by definition.

$$C_n^k = \binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-1}b^2 \cdots + \binom{n}{n} b^n.$$

In order to verify this well-known theorem, let us expand

$$P = (a+b_1)(a+b_2)\dots(a+b_n).$$

$$P = a^n + B_1 a^{n-1} + B_2 a^{n-2} + \cdots + B_{n-1}a + B_n$$

here

$$B_1 = b_1 + b_2 + \cdots + b_n,$$

$$B_2 = b_1 b_2 + b_1 b_3 + \cdots + b_1 b_n + b_2 b_3 + \cdots + b_{n-1} b_n$$

$$\vdots$$

$$B_n = b_1 b_2 b_3 \dots b_1 b_n.$$

Set

$$b_1 = b_2 = \cdots = b_n = b.$$

Then

$$B_k = \binom{n}{k} b^k.$$

To conclude, substitute B_k to P.

Lemma on $\sqrt{n}$

Let n be a positive integer. If $\sqrt{n}$ is not an integer then it is irrational.

Proof:

If $\sqrt{n}$ is an integer, we have nothing to prove. Suppose, if possible, that $\sqrt{n}$ is a rational number

$$\sqrt{n} = \frac{p}{q},$$

where p and q are positive integers and relative primes. Then

$$n = \frac{p^2}{q^2},$$

and

$$nq^2 = p^2.$$

Let r be a factor in the unique prime decomposition of n with exponent α, α odd. Such r exists because if each prime factor of n has even exponent then $\sqrt{n}$ is integer, contrary to our assumption.

If prime r divides p^2 then p divides p. Consequently, r has an even exponent in p^2. Contradiction, because on the other side of the equation r is on odd power. Therefore n is not rational, not integer, but an irrational number.

Corollary:

$\sqrt{3}$ is irrational.

Corollary:

If $n > 0$ is not a k-th power of an integer then the k-th root of n is an irrational number.

Proof of Theorem:

Write $x = \sqrt{3}$. Then $x^2 = 3$.

Claim I.

$$\left[(1+x)^{2m+1}\right] = (1+x)^{2m+1} + (1-x)^{2m+1}.$$

Proof:

$$(1+x)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} 1^k x^{2m+1-k}$$

$$(1-x)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} 1^k (-x)^{2m+1-k}$$

If k is odd then $2m+1-k$ is even and

$$\binom{2m+1}{k} 1^k x^{2m+1-k} = \binom{2m+1}{k} 1^k (-x)^{2m+1-k};$$

if k is even then $2m+1-k$ is odd and

$$\binom{2m+1}{k} 1^k x^{2m+1-k} = -\binom{2m+1}{k} 1^k (-x)^{2m+1-k}.$$

After cancellation the integer summands yield

$$[(1+x)^{2m+1} + (1-x)^{2m+1}] = 2 \sum_{0\le k\le 2m+1}^{k\ even} \binom{2m+1}{k} 1^k x^{2m+1-k} \text{ integer.}$$

On the other hand , $0 < (1-x)^{2m+1} < 1$

$$(1+x)^{2m+1} - 1 < (1+x)^{2m+1} + (1-x)^{2m+1} < (1+x)^{2m+1}$$

Thus $(1+x)^{2m+1} + (1-x)^{2m+1}$ is an integer between $(1+x)^{2m+1} - 1$ and $(1+x)^{2m+1}$ hence it is equal to $[(1+x)^{2m+1}]$.

Claim II.

$$\left[(1+x)^{2m+1}\right] = 2^m\{(1+x)(2+x)^m + (1-x)(2-x)^m$$

Proof:

$$(1+x)^{2m+1} = (1+x)((1+x)^2)^m = (1+x)(4+2x)^m = 2^m(1+x)(2+x)^m$$
$$(1-x)^{2m+1} = (1-x)((1-x)^2)^m = (1-x)(4-2x)^m = 2^m(1-x)(2-x)^m$$

Summing these two lines gives

$$(1+x)^{2m+1} + (1-x)^{2m+1} = 2^m\{(1+x)(2+x)^m + (1-x)(2-x)^m\}.$$

Claim III.

$$\{(1+x)(2+x)^m + (1-x)(2-x)^m\} =$$
$$2(a+bx) + (1+x)x^m + 2(a-bx) + (1-x)(-x)^m,$$

where a and b are integers.

Proof:

$$g(x) = (2{+}x)^m = \begin{pmatrix} m \\ 0 \end{pmatrix} 2^m {+} \begin{pmatrix} m \\ 1 \end{pmatrix} 2^{m-1}x {+} \begin{pmatrix} m \\ 2 \end{pmatrix} 2^{m-2}x^2 \cdots {+} \begin{pmatrix} m \\ m \end{pmatrix} x^m.$$

$$g(x) = c_0 + c_1x + c_2x^2 + \cdots + c_mx^m$$

where

$$c_k = \begin{pmatrix} m \\ k \end{pmatrix} 2^{m-k}, \quad k = 0, 1, 2 \ldots m$$

$$(1+x)g(x) =$$

$$c_0 + (c_0 + c_1)x + (c_1 + c_2)x^2 + \cdots + (c_{m-1} + c_m)x^m + c_mx^{m+1} =$$

$$c_0 + (c_0 + c_1)x + (c_1 + c_2)x^2 + \cdots + c_{m-1}x^m + c_mx^m + c_mx^{m+1} =$$

$$c_0 + (c_0 + c_1)x + (c_1 + c_2)x^2 + \cdots + c_{m-1}x^m + (1+x)x^m =$$

$$c_0 {+} (c_0 {+} c_1)x {+} (c_1 {+} c_2)x^2 {+} \cdots {+} (c_{m-2} {+} c_{m-1})x^{m-1} {+} c_{m-1}x^m {+} (1{+}x)x^m =$$

$$\big(c_0 + (c_1 + c_2)x^2 + (c_3 + c_4)x^4 + \ldots\big) {+} \big((c_0 + c_1)x + (c_2 + c_3)x^3 + \ldots\big) {+} (1{+}x)x^m.$$

$$\big(c_0 + (c_1 + c_2)x^2 + (c_3 + c_4)x^4 + \ldots\big) {+} \big((c_0 + c_1) + (c_2 + c_3)x^2 + \ldots\big)\, x {+} (1{+}x)x^m.$$

Further

$$g(-x) = (2{-}x)^m = \begin{pmatrix} m \\ 0 \end{pmatrix} 2^m {-} \begin{pmatrix} m \\ 1 \end{pmatrix} 2^{m-1}x {+} \begin{pmatrix} m \\ 2 \end{pmatrix} 2^{m-2}x^2 \cdots {\pm} \begin{pmatrix} m \\ m \end{pmatrix} x^m.$$

$$g(-x) = c_0 - c_1x + c_2x^2 \ \cdots \pm c_mx^m$$

$$(1{-}x)g(-x) = \big[c_0 - c_1x + c_2x^2 \ \cdots \pm c_mx^m\big] {-} \big[c_0 - c_1x + c_2x^2 \ \cdots \pm c_mx^m\big]\, x =$$

$$\big(c_0 + (c_1 + c_2)x^2 + (c_3 + c_4)x^4 + \ldots\big) {-} \big((c_0 + c_1) + (c_2 + c_3)x^2 + \ldots\big)\, x {+} (1{-}x)(-x)^m.$$

$$(1-x)g(-x) = 2(a - bx) + (1-x)(-x)^m$$

Case I, m odd, :

$$(1+x)g(x)+(1-x)g(-x)=4a+2x^{m+1}=2(2a+x^{m+1}).$$

First, collect terms with $x^k,\;\; k=0,1,2,\ldots,m-1$

$$2a=c_0+[c_1+c_2]x^2+[c_3+c_4]x^4+\cdots+[c_{m-2}+c_{m-1}]x^{m-1};$$

in concrete form

$$a=\begin{pmatrix} m \\ 0 \end{pmatrix}2^{m-1}x^0+\left[\begin{pmatrix} m \\ 1 \end{pmatrix}2^{m-2}+\begin{pmatrix} m \\ 2 \end{pmatrix}2^{m-3}\right]x^2+\ldots$$

$$+\left[\begin{pmatrix} m \\ m-2 \end{pmatrix}2+\begin{pmatrix} m \\ m-1 \end{pmatrix}\right]x^{m-1},$$

since x^{2k} is a power of 3, a is an odd integer.

Next, collect terms with $x^{2k+1},\;\; k=0,1,2,\ldots,m$

$$2bx=\left[\begin{pmatrix} m \\ 0 \end{pmatrix}2^{m}+\begin{pmatrix} m \\ 1 \end{pmatrix}2^{m-1}\right]x^1+\left[\begin{pmatrix} m \\ 2 \end{pmatrix}2^{m-2}+\begin{pmatrix} m \\ 3 \end{pmatrix}2^{m-3}\right]x^3+\ldots$$

$$+\left[\begin{pmatrix} m \\ m-3 \end{pmatrix}2^3+\begin{pmatrix} m \\ m-2 \end{pmatrix}2^2\right]x^{m-2}+\begin{pmatrix} m \\ m-1 \end{pmatrix}2^1x^m,$$

then

$$b=\left[\begin{pmatrix} m \\ 0 \end{pmatrix}2^{m-1}+\begin{pmatrix} m \\ 1 \end{pmatrix}2^{m-2}\right]x^0+\left[\begin{pmatrix} m \\ 2 \end{pmatrix}2^{m-3}+\begin{pmatrix} m \\ 3 \end{pmatrix}2^{m-4}\right]x^2+\ldots$$

$$+\left[\begin{pmatrix} m \\ m-3 \end{pmatrix}2^2+\begin{pmatrix} m \\ m-2 \end{pmatrix}2^1\right]x^{m-3}+\begin{pmatrix} m \\ m-1 \end{pmatrix}2^0x^{m-1},$$

and thirdly,

$$c_mx^m+c_mx^{m+1}=(1+x)x^m.$$

$$(1+x)g(x)=(1+x)(2+x)^m==2(a+bx)+(1+x)x^m.$$

$$(1-x)g(-x)=(1-x)(2-x)^m=2(a-bx)+(1-x)(-x)^m$$

$$(1+x)g(x)+(1-x)g(-x)=2(a+bx)+(1+x)x^m+2(a-bx)+(1-x)(-x)^m$$

$$(1+x)g(x)+(1-x)g(-x)=2a+x^m+x^{m+1}+2a-x^m+x^{m+1}$$

$$(1+x)g(x)+(1-x)g(-x)=4a+2x^{m+1}=2(2a+x^{m+1}).$$

Claim IV.

$(2a+x^{m+1})$ is odd, not divisible by 2.

$\left[(1+\sqrt{x})^{2m+1}\right] = 2^m\{(1+x)(2+x)^m+(1-x)(2-x)^m == 2^{m+1}(2a+x^{m+1})$.

End of Proof of Case I.

Demonstration for m=5

Claim V.

$$\left[(1+x)^{11}\right] = (1+x)^{11} + (1-x)^{11}.$$

Proof:

$$(1+x)^{11} = \sum_{k=0}^{11} \binom{11}{k} 1^k x^{11-k}$$

$$(1-x)^{11} = \sum_{k=0}^{11} \binom{11}{k} 1^k (-x)^{11-k}$$

If k is odd then $11-k$ is even and

$$\binom{11}{k} 1^k x^{11-k} = \binom{11}{k} 1^k (-x)^{11-k};$$

if k is even then $11-k$ is odd and

$$\binom{11}{k} 1^k x^{11-k} = -\binom{11}{k} 1^k (-x)^{11-k}.$$

After cancelation the integer summands yield

$$[(1+x)^{11} + (1-x)^{11}] = 2 \sum_{0\le k\le 11}^{k\ even} \binom{11}{k} 1^k x^{11-k} \ \text{ integer.}$$

On the other hand , $0 < (1-x)^{11} < 1$

$$(1+x)^{11} - 1 < (1+x)^{11} + (1-x)^{11} < (1+x)^{11}$$

Thus $(1+x)^{11} + (1-x)^{11}$ is an integer between $(1+x)^{11} - 1$ and $(1+x)^{11}$ hence it is equal to $[(1+x)^{11}]$.

Claim VI.

$$(1+x)^{11} = (1+x)((1+x)^2)^5 = (1+x)(4+2x)^5 = 2^5(1+x)(2+x)^5$$
$$(1-x)^{11} = (1-x)((1-x)^2)^5 = (1-x)(4-2x)^5 = 2^5(1-x)(2-x)^5$$
$$(1+x)^{11} + (1-x)^{11} = 2^5\{(1+x)(2+x)^5 + (1-x)(2-x)^5\}$$
$$\left[(1+\sqrt{x})^{11}\right] = 2^5\{(1+x)(2+x)^5 + (1-x)(2-x)^5\}.$$

Claim VII.

$$\{(1+x)(2+x)^5 + (1-x)(2-x)^5\} =$$
$$2(a+bx) + (1+x)x^5 + 2(a-bx) + (1-x)(-x)^5.$$

$$(1+x)(2+x)^5 =$$

$$g(x) = (2+x)^m =$$
$$\binom{5}{0} 2^5 x^0$$

$$+\left[\binom{5}{0}2^5 + \binom{5}{1}2^4\right]x^1$$

$$+\left[\binom{5}{1}2^4 + \binom{5}{2}2^3\right]x^2$$

$$+\left[\binom{5}{2}2^3 + \binom{5}{3}2^2\right]x^3$$

$$+\left[\binom{5}{3}2^2 + \binom{5}{4}2^1\right]x^4$$

$$+\left[\binom{5}{4}2^1 + \binom{5}{5}2^0\right]x^5$$

$$+\binom{5}{5}2^0x^6.$$

$$(1+x)(2+x)^5 = x^6 + 11x^5 + 50x^4 + 120x^3 + 160x^2 + 112x + 32$$

Mutatis mutandis

$$-x \to x$$

$$(1-x)(2-x)^5 =$$

$$\binom{5}{0} 2^5 x^0$$

$$-\left[\binom{5}{0} 2^5 + \binom{5}{1} 2^4\right] x^1$$

$$+\left[\binom{5}{1} 2^4 + \binom{5}{2} 2^3\right] x^2$$

$$-\left[\binom{5}{2} 2^3 + \binom{5}{3} 2^2\right] x^3$$

$$+\left[\binom{5}{3} 2^2 + \binom{5}{4} 2^1\right] x^4$$

$$-\left[\binom{5}{4} 2^1 + \binom{5}{5} 2^0\right] x^5$$

$$+\binom{5}{5} 2^0 x^6.$$

Let us rearrange the sums

$$(1+x)(2+x)^5 =$$

$$\binom{5}{0} 2^5 x^0 + \left[\binom{5}{1} 2^4 + \binom{5}{2} 2^3\right] x^2 + \left[\binom{5}{3} 2^2 + \binom{5}{4} 2^1\right] x^4 + \binom{5}{5} 2^0 x^6$$

$$+\left[\binom{5}{0} 2^5 + \binom{5}{1} 2^4\right] x^1 + \left[\binom{5}{2} 2^3 + \binom{5}{3} 2^2\right] x^3 + \left[\binom{5}{4} 2^1 + \binom{5}{5} 2^0\right] x^5 =$$

$$\binom{5}{0}2^5x^0+\left[\binom{5}{1}2^4+\binom{5}{2}2^3\right]x^2+\left[\binom{5}{3}2^2+\binom{5}{4}2^1\right]x^4$$
$$+\left[\binom{5}{0}2^5+\binom{5}{1}2^4\right]x^1+\left[\binom{5}{2}2^3+\binom{5}{3}2^2\right]x^3+\binom{5}{4}2^1x^5$$
$$(1+x)x^5.$$
$$(1+x)(2+x)^5 = 2(a+bx)+(1+x)x^5$$

where

$$2a=\binom{5}{0}2^5x^0+\left[\binom{5}{1}2^4+\binom{5}{2}2^3\right]x^2+\left[\binom{5}{3}2^2+\binom{5}{4}2^1\right]x^4$$

$$2bx=\left[\binom{5}{0}2^5+\binom{5}{1}2^4\right]x^1+\left[\binom{5}{2}2^3+\binom{5}{3}2^2\right]x^3+\binom{5}{4}2^1x^5$$

in concrete form

$$a=\binom{5}{0}2^4x^0+\left[\binom{5}{1}2^3+\binom{5}{2}2^2\right]x^2+\left[\binom{5}{3}2^1+\binom{5}{4}2^0\right]x^4$$

$$b=\left[\binom{5}{0}2^4+\binom{5}{1}2^3\right]x^0+\left[\binom{5}{2}2^2+\binom{5}{3}2^1\right]x^2+\binom{5}{4}2^0x^4$$
$$(1+x)(2+x)^5 = x^6+11x^5+50x^4+120x^3+160x^2+112x+32$$
$$a=25x^4+80x^2+16$$
$$b=5x^4+60x^2+56$$
$$(1+x)(2+x)^5 = 2(a+bx)+(1+x)x^5. \ \surd$$

Note, that both a and b are integers.

Similar steps lead from

$$(1-x)(2-x)^5 = x^6-11x^5+50x^4-120x^3+160x^2-112x+32$$

to

$$(1-x)(2-x)^5 = 2(a-bx)+(1-x)(-x)^5. \ \surd$$

where a and b are as defined above.

Calculation:

$$(1+\sqrt{3})^{11} = 63296.032356$$

$$[](1+\sqrt{3})^{11}] = 63296$$

$$64 = 2^6, \quad 63296 : 64 = 989.$$

The highest power of 2 that divides 63296 is 2^6.

Case II, m even :

Same method applies.

Demonstration for m=4

Claim 0.

$$2m+1 = 9$$

$$(1+\sqrt{3})^9 = 8480.06037692\ldots$$

$$[(1+\sqrt{3})^9] = 8480$$

$$32 = 2^5, \quad 8480 : 32 = 265$$

The highest power of 2 that divides 8480 is 2^5.

Claim I.

$$\left[(1+x)^9\right] = (1+x)^9 + (1-x)^9.$$

Proof:

$$(1+x)^9 = \sum_{k=0}^{9} \binom{9}{k} 1^k x^{9-k}$$

$$(1-x)^9 = \sum_{k=0}^{9} \binom{9}{k} 1^k (-x)^{9-k}$$

where

$$(1-x)^9 = (1+(-x))^9; \quad \text{binom theorem expansion.}$$

If k is odd then $9-k$ is even and

$$\binom{9}{k}1^k x^{9-k} = \binom{9}{k}1^k(-x)^{9-k};$$

if k is even then $9-k$ is odd and

$$\binom{9}{k}1^k x^{9-k} = -\binom{9}{k}1^k(-x)^{9-k}.$$

After cancellation the integer summands yield

$$[(1+x)^9 + (1-x)^9] = 2\sum_{0\le k\le 9}^{k\ even}\binom{9}{k}1^k x^{9-k} \quad \text{integer.}$$

On the other hand , $0 < (1-x)^9 < 1$

$$(1+x)^9 - 1 < (1+x)^9 + (1-x)^9 < (1+x)^9$$

Thus $(1+x)^9+(1-x)^9$ is an integer between $(1+x)^{19}-1$ and $(1+x)^9$ hence it is equal to $[(1+x)^9]$, same as before.

Claim II.

When

$$x = \sqrt{3}$$

we have

$$\begin{aligned}(1+x)^9 &= (1+x)((1+x)^2)^4 = (1+x)(1+2x+x^2)^4 \\ &= (1+x)(4+2x)^4 \\ &= 2^4(1+x)(2+x)^4,\end{aligned}$$

Thus two polynomials of different orders are equal at point

$$(1+x)^9 = 2^4(1+x)(2+x)^4, \text{if } \ x=\sqrt{3}.$$

Check:

$$(1+x)^9 = 4240 + 2448\sqrt{3}$$

$$2^4(1+x)(2+x)^4 = 4240 + 2448\sqrt{3}.\surd$$

Similarly

$$\begin{aligned}(1-x)^9 &= (1-x)((1-x)^2)^4 \\ &= (1-x)(1-2x+x^2)^4 \\ &= (1-x)(4-2x)^4 \\ &= 2^4(1-x)(2-x)^4\end{aligned}$$

$$(1-x)^9 = 4240 - 2448\sqrt{3}$$

$$2^4(1-x)(2-x)^4 = 4240 - 2448\sqrt{3}\ \surd.$$

$$(1+x)^9 + (1-x)^9 = 2^4\{(1+x)(2+x)^4 + (1-x)(2-x)^4\}$$

$$\left[(1+x)^9\right] = 2^4\{(1+x)(2+x)^4 + (1-x)(2-x)^4\}.$$

Thus the integer part of $(1+x)^9$ at $x = \sqrt{3}$ is equal to the sum of two polynomials of lower order, $2^4(1+x)(2+x)^4,\ 2^4(1-x)(2-x)^4$ evaluated at the same point, $x = \sqrt{3}$.

Claim III.

Write

$$\begin{aligned}g(x) &= (2+x)^4 \\ &= \sum_{k=0}^{k+4} \binom{4}{k} 2^k x^{4-k} \\ &= c_0x^4 + c_1x^3 + c_2x^2 + c_3x + c_4\end{aligned}$$

where

$$c_k = \binom{4}{k} 2^k, \quad k = 0, 1, \ldots 4.$$

$$(1+x)g(x) = c_0x^5 + (c_0+c_1)x^4 + (c_1+c_2)x^3 + (c_2+c_3)x^2 + (c_3+c_4)x + c_4$$

Since $c_0 = 1$,

$$\begin{aligned}(1+x)g(x) &= \left[c_1x^4 + (c_2+c_3)x^2 + c_4\right] \\ &+ \left[(c_1+c_2)x^3 + (c_3+c_4)x\right] \\ &+ \left[(1+x)x^4\right]\end{aligned}$$

$$\begin{aligned} 2a &= \left[c_1x^4 + (c_2 + c_3)x^2 + c_4\right] \\ 2bx &= \left[(c_1 + c_2)x^2 + (c_3 + c_4)\right] x \\ (1 + x)g(x) &= 2(a + bx) + (1 + x)x^4 \end{aligned}$$

Calculate a, b.

$$2a = [8 * 9 + (24 + 32) * 3 + 16] = 256$$

$$2b = [(8 + 24) * 3 + (32 + 16)] = 144$$

Evaluate both sides at $x = \sqrt{3}$

$$2(a + bx) + (1 + x)x^4 = 2(128 + 72x) + (1 + x) * 9 = 265 + 153x$$

$$(1 + x)g(x) = 265 + 153x, \ \surd$$

Tis proves that

$$(1 + x)g(x) = 2(a + bx) + (1 + x)x^4.$$

Also,

$$(1 - x)g(-x) = 2(a - bx) + (1 - x)(-x)^4$$

because

$$(1 - x)g(-x) = 265 - 153x$$

$$2(a - bx) + (1 - x)(-x)^4 = 265 - 153x, \ \surd$$

Then

$$\{(1 + x)(2 + x)^4 + (1 - x)(2 - x)^4\} = (1 + x)g(x) + (1 - x)g(-x).$$

Furthermore, since

$$(1 + x)g(x) = 2(a + bx) + (1 + x)x^4$$

$$(1 - x)g(-x) = 2(a - bx) + (1 - x)(-x)^4$$

$$\begin{aligned} (1 + x)g(x) + (1 - x)g(-x) &= 2(a + bx) + (1 + x)x^4 + 2(a - bx) + (1 - x)(-x)^4 \\ &= 4a + 2x \\ &= 2(2a + x). \end{aligned}$$

Note that $(2a + x)$ is an odd integer.

La Grande Finale:

$$\left[(1 + x)^9\right] = 2^m\{2(2a + x)\}$$

where a, m, x are as defined above.

20.2 Assignment 49. Revised

- Problem-solving Seminar
- *Ciarlet et al.: Exercises d'analyse numérique matricielle,*
- Last revision April 22, 2020

Problems

1.1.-1. *Let A be an invertible matrix whose elements, as well as those of A^{-1} are non-negative. Show that there exists a permutation matrix P and a matrix $D = diag(d_i)$, with d_i positive , such that $A = PD$ (converse is obvious).*

Discussion I.: Let us consider a 4×4 matrix A. By construction, A is invertible, thus $|A| \neq 0$. The determinant of A is a sum of products, each product comprises a diagonal; which has four elements, in this example, one from each row and each column. If every product is zero then the determinant is zero and A is not invertible. That is absurd. Therefore at least one diagonal is not zero. For demonstration, let us suppose that

$$a_{13}a_{24}a_{32}a_{41} \neq 0;$$

where

$$a_{13} > 0; \; a_{24} > 0; \; a_{32} > 0; \; a_{41} > 0.$$

Then

$$A = \begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{bmatrix}$$

is an invertible matrix with non-negative elements

$$|A| = a_{13}a_{24}a_{32}a_{41} > 0.$$

Next, we find B, the inverse pair of A by costructing the adjoint first. Recall that the adjoint of A is the transposed matrix of cofactors of A and

$$B = \frac{1}{detA} adjA$$

is the inverse of A. The non-zero elements of B are calculated as follows:

$$b_{14} = \frac{a_{13}a_{24}a_{32}}{a_{13}a_{24}a_{32}a_{41}} = \frac{1}{a_{41}}$$

$$b_{23} = \frac{a_{13}a_{24}a_{41}}{a_{13}a_{24}a_{32}a_{41}} = \frac{1}{a_{32}}$$

$$b_{31} = \frac{a_{24}a_{32}a_{41}}{a_{13}a_{24}a_{32}a_{41}} = \frac{1}{a_{13}}$$

$$b_{42} = \frac{a_{13}a_{32}a_{41}}{a_{13}a_{24}a_{32}a_{41}} = \frac{1}{a_{24}}$$

$$\begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} a_{13}b_{31} & 0 & 0 & 0 \\ 0 & a_{24}b_{42} & 0 & 0 \\ 0 & 0 & a_{32}b_{23} & 0 \\ 0 & 0 & 0 & a_{41}b_{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore if matrix A has one and only one diagonal with positive elements and all other elements are zero then A is invertible and its inverse B has one and only one diagonal with positive elements and all other elements of B are zero.

Discussion II.: So far we have obtained a pair of matrices A, B non-negative, $AB = I$. Write

$$A = \begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = PD$$

Verification:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \\ 0 & d_2 & 0 & 0 \\ d_1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{bmatrix}.$$

P is a permutation matrix, D is a diagonal matrix, moreover $B = A^{-1}$.

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix} = \begin{bmatrix} d_1^{-1} & 0 & 0 & 0 \\ 0 & d_2^{-1} & 0 & 0 \\ 0 & 0 & d_3^{-1} & 0 \\ 0 & 0 & 0 & d_4^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = D^{-1}P^{-1}.$$

Verification:

$$\begin{bmatrix} d_1^{-1} & 0 & 0 & 0 \\ 0 & d_2^{-1} & 0 & 0 \\ 0 & 0 & d_3^{-1} & 0 \\ 0 & 0 & 0 & d_4^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & d_1^{-1} \\ 0 & 0 & d_2^{-1} & 0 \\ d_3^{-1} & 0 & 0 & 0 \\ 0 & d_4^{-1} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix}.$$

Both A and B can be written as a product of a permutation matrix and a diagonal matrix. This demonstrates the "converse" or the "obvious" part of the theorem.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad P^{-1} = P^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

We can formulate a tentative theorem: Let $D = diag(d_i)$ be an $n \times n$ matrix with $\forall d_i > 0$ and P be a permutation matrix of the same order. Write

$$D^{-1} = diag(\frac{1}{d_i}),\ A = PD,\ B = D^{-1}P^T,\ P^{-1} = P^T.$$

Then A, B are non-negative and

$$AB = I.$$

However, $\forall d_i > 0$ can be replaced by $\forall d_i \neq 0$. Here is a numerical example:

$$\begin{bmatrix} 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{inversion} \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Discussion III.: Suppose, if possible, that matrix A' has a diagonal - say the same as A - plus one positive element y and its inverse is all non-negative.

$$A' = \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & x & 0 & 0 \\ x & 0 & y & 0 \end{bmatrix}.$$

Let B' be the inverse of A', assuming that all elements of B' are non-negative.

$$B' = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$A'B' = \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & x & 0 & 0 \\ x & 0 & y & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} =$$

$$\begin{bmatrix} xb_{31} & xb_{32} & xb_{33} & xb_{34} \\ xb_{41} & xb_{42} & xb_{43} & xb_{44} \\ xb_{21} & xb_{22} & xb_{23} & xb_{24} \\ xb_{11} + yb_{31} & xb_{12} + yb_{32} & xb_{13} + yb_{33} & xb_{14} + yb_{34} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There is no cancellation because x, y are positive and b_{ij} are all non-negative, hence

$$xb_{32} = xb_{33} = xb_{34} = 0$$

$$xb_{41} = xb_{43} = xb_{44} = 0$$

$$xb_{21} = xb_{22} = xb_{24} = 0$$

$$xb_{11} + yb_{31} = xb_{12} + yb_{32} = xb_{13} + yb_{33} = 0$$

But then

$$b_{11} = b_{21} = b_{31} = b_{41} = 0$$

one column consists of all zeros, B' is singular, or non-invertible. But this is absurd. Therefore there is no non-zero element y,and matrix A' has a single diagonal. This, of course is not a proof, it is only a demonstration of the main idea of the proof.

Discussion IV.: Sherman-Morison Formula states

$$(A + uv^T) = A^{-1} - \frac{(A^{-1}u)(v^T A^{-1})}{1 + v^T A^{-1} u},$$

where matrix A is invertible, u, v are vectors, and $1 + v^T A^{-1} u \neq 0$. Write

$$u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 0 \\ y \\ 0 \end{bmatrix}, \ y > 0;\ uv^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{bmatrix}.$$

Then

$$(A + uv^T) = A'$$

as before. Next, we shall calculate the inverse of A'.

Let

$$A^{-1}u = \begin{bmatrix} 0 & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_{14} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$v^T A^{-1} = \begin{bmatrix} 0 & 0 & y & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix} = \begin{bmatrix} yb_{31} & 0 & 0 & 0 \end{bmatrix}$$

Diadic product of

$$\begin{bmatrix} b_{14} \\ 0 \\ 0 \\ 0 \end{bmatrix} \bigotimes \begin{bmatrix} yb_{31} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} y * b_{31} * b_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad y * b_{31} * b_{14} > 0.$$

Inner product of the same vectors is $v^T A^{-1} u = y * b_{31} * b_{14} > 0$. Substitution yields

$$(A + uv^T)^{-1} = A^{-1} - \begin{bmatrix} \frac{y * b_{31} * b_{14}}{1 + y * b_{31} * b_{14}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and finally

$$(A + uv^T)^{-1} = \begin{bmatrix} -\frac{y * b_{31} * b_{14}}{1 + y * b_{31} * b_{14}} & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix}$$

This shows that one element $(1, 1)$ is negative. which contradicts the assumption that the inverse has only non-negative elements. Therefore there cannot be an extra element y.

Note, that $(A + uv^T)^{-1})$ is not a diagonal. Therefore $(A + uv^T)^{-1}$ cannot be a diagonal either. so there is at least one element not on A^{-1}, in the above example it is element $(1, 1)$.

Review of Sherman - Morrison Formula

$$(A + uv^T)^{-1} = A^{-1} - \frac{(A^{-1}u)(v^T A^{-1})}{1 + v^T A^{-1} u}$$

Discussion:

A is an invertible $n \times n$ matrix, u, v are compatible column vectors. Suppose A is modified by a diadic product and the new matrix is $(A + uv^T)$. The formula provides an inverse for the new matrix $(A + uv^T)$. $(A^{-1}u)$ is column vector, the product $(v^T A^{-1})$ is a row vector. $(A^{-1}u)(v^T A^{-1})$ is a diadic product, an $n \times n$ matrix. $(1 + v^T A^{-1} u)$ is a (non-zero) scalar, $(v^T A^{-1} u)$ is a scalar, too.

Verification:

$$i) \quad (A + uv^T)\left(A^{-1} - \frac{(A^{-1}u)(v^T A^{-1})}{1 + v^T A^{-1} u}\right) =$$

$$AA^{-1} + uv^T A^{-1} - \left(\frac{AA^{-1}uv^T A^{-1} + uv^T A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u}\right) =$$

$$I + uv^T A^{-1} - \left(\frac{uv^T A^{-1} + uv^T A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u}\right) =$$

$$I + uv^T A^{-1} - \frac{u(v^T A^{-1} + v^T A^{-1} uv^T A^{-1})}{1 + v^T A^{-1} u} =$$

$$I + uv^T A^{-1} - \frac{u(v^T A^{-1} + v^T A^{-1} uv^T A^{-1})}{1 + v^T A^{-1} u} =$$

$$I + uv^T A^{-1} - \frac{u(1 + v^T A^{-1} u)v^T A^{-1}}{1 + v^T A^{-1} u} =$$

$$I + uv^T A^{-1} - uv^T A^{-1} = I. \surd$$

$$ii) \quad \left(A^{-1} - \frac{(A^{-1}u)(v^T A^{-1})}{1 + v^T A^{-1} u}\right)(A + uv^T) =$$

$$A^{-1}A + A^{-1}uv^T - \frac{A^{-1}uv^TA^{-1}A + A^{-1}uv^TA^{-1}uv^T}{1 + v^TA^{-1}u} =$$
$$I + A^{-1}uv^T - \frac{A^{-1}u(1 + v^TA^{-1}u)v^T}{1 + v^TA^{-1}u} =$$
$$I + A^{-1}uv^T - A^{-1}uv^T = I.\surd$$

$iii)\ \ 1 + v^TA^{-1}u = 0$

$x = A^{-1}u \neq 0$

$1 + v^Tx = 0$

$u + uv^Tx = 0$

$Ax + uv^Tx = 0$

$(A + uv^T)x = 0.$

Therefore if $1 + v^TA^{-1}u = 0$ then $\ker(A + uv^T) \neq \emptyset$ hence it is singular, and does not have and inverse.

1.1.-2. *Let $A = [a_{i,j}]$ be a matrix of type (m, n). Show that*

$$\max_i\{\min_j a_{ij}\} \leq \min_j\{\max_i a_{ij}\}$$

Proof:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Write y_i for the greatest element in the i-th row and z_j for the least element in the j-th column. Thus

$$y_i \geq a_{ij} \geq z_j, \quad \forall i, j$$

Consequently,

$$\min_i y_i \geq z_j, \quad \forall i, j$$

and

$$\min_i y_i \geq \max_j z_j, \quad \forall i, j$$

1.1.-3. *Let A and B be two square matrices of the same order. Show that the matrices AB and BA have the same characteristic polynomial.*

Observation: It is easy to see that matrices AB and BA have the same *minimal polynomial* .From

$$AB = BA$$

it follows that

$$(AB)^m = (BA)^m, \ m = 0, 1, 2, \ldots$$

Suppose

$$f(x) = x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x + a_0$$

is a monic polynomial, $\forall a_k$ real. If monic matrix polynomial

$$f(AB) = (AB)^k + a_{k-1}(AB)^{k-1} + a_{k-2}(AB)^{k-2} + \cdots + a_1(AB) + a_0 I = 0,$$

the zero matrix of same order then $f(AB)$ is an *annihilating polynomial* of matrix AB. By the equality of $(AB)^m = (BA)^m$

$$f(BA) = (BA)^k + a_{k-1}(BA)^{k-1} + a_{k-2}(BA)^{k-2} + \cdots + a_1(BA) + a_0 I = 0,$$

is an annihilating polynomial of matrix BA as well, and *vice versa* . Therefore the set of annihilating polynomials of AB is precisely the set of annihilating polynomials of BA. The unique monic generator of the ideal of all polynomials that annihilate AB, hence BA, is called the *minimal polynomial* . The characteristic and minimal polynomials of AB (and BA) have the same roots, except for multiplicities.

Therefore the characteristic polynomials of AB and BA have the same spectrum.

Proof I: As before, A, B, I are square matrices of the same order. Suppose $|A| \neq 0$, matrix A is invertible.

$$|A||A^{-1}| = 1$$

$$|BA - \lambda I| = |A^{-1}(AB - \lambda I)A| = |A^{-1}||(AB - \lambda I)|A| = |(AB - \lambda I)|$$

and the characteristic polynomials are equal.

Next, write

$$P = I - AB, \quad Q = I - BA$$

not assuming $|A| \neq 0$. We claim that if P^{-1} exists so does Q^{-1}.

$$\begin{aligned} QBP^{-1}A &= (I - BA)B(I - AB)^{-1}A \\ &= (B - BAB)(I - AB)^{-1}A \\ &= B(I - AB)(I - AB)^{-1}A \\ &= BA \end{aligned}$$

$$Q + QBP^{-1}A = Q(I + BP^{-1}A) = Q + BA = I$$

$$Q^{-1} = I + BP^{-1}A$$

Suppose now that $\lambda \neq 0$ is not an eigenvalue of AB. Then $|\lambda I - AB| \neq 0$.

$$|\lambda I - AB| \neq 0 \quad \Rightarrow |I - \frac{1}{\sqrt{\lambda}}A\frac{1}{\sqrt{\lambda}}B| \neq 0 \quad \Rightarrow$$

by the foregoing argument

$$|I - \frac{1}{\sqrt{\lambda}}B\frac{1}{\sqrt{\lambda}}A| \neq 0 \quad \Rightarrow |\lambda I - BA| \neq 0$$

Thus λ is not an eigenvalue of BA.

Let $\lambda = 0$, be not an eigenvalue of AB.

$$|\lambda I - AB| \neq 0 \quad \Rightarrow |AB| \neq 0 \quad \Rightarrow |BA| \neq 0.$$

Therefore if λ is not an eigenvalue of AB then it is not an eigenvalue of BA. This proves the theorem.

Proof II: Let us recall, that A and B square matrices are mappings from an underlying n-dimensional vector space to itself. A vector x is said to be an *eigenvector* of A if $x \neq 0$ and for some scalar λ Matrix A as a mapping of a finite-dimensional vectorspace onto itself over an algebraically closed field has at least one eigenvalue and a subspace associated with it which coincides with $Ker(A - \lambda I)$.

$$Ax = \lambda x.$$

Let

$$BAy = \lambda y.$$

Write

$$B(Ay) = \lambda y$$

by the associative property of multiplication.

$$AB(Ay) = \lambda Ay$$

Thus if $Ay \neq 0$ then λ is an eigenvalue of BA as well as AB. If, on the other hand, $Ay = 0$, then $\lambda = 0$ This shows that all the non-zero eigenvalues of BA are that of AB, and *vice versa* , of course. Moreover, if $Ay \neq 0$, $\lambda = 0$ then λ is an eigenvalue of BA as well as AB.

It remains to check $Ay = 0$, $y \neq 0$. Then the null space of A is not empty, consequently the null space of BA is not empty, either. Hence there exists a vector, say z, such that

$$BAz = 0,$$

and $\lambda = 0$ is also an eigenvalue of BA.

1.1.-4. *Let a, b, and c be given scalars. find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} a & b & 0 & 0 & \ldots & 0 \\ c & a & b & 0 & \ldots & \ldots \\ 0 & c & a & b & 0 & \ldots \\ 0 & 0 & c & a & b & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & c & a \end{bmatrix}$$

Discussion: This problem - like many other Problems in Ciarlet's collection - has well-known, published solution. For example, *Rózsa* gives a complete, elegant solution using Lagrange matrix-polynomials. We trace a different solution. The matrix in question is the common tridiagonal matrix, the eigenvalues are

$$\lambda_s = a + 2\sqrt{bc}\cos\frac{s\pi}{n+1} \quad s = 1, 2, \ldots n.$$

Once the eigenvalues are determined the coordinates of eigenvectors can be calculated recursively from

$$\begin{bmatrix} a-\lambda & b & 0 & 0 & \ldots & 0 \\ c & a-\lambda & b & 0 & \ldots & \ldots \\ 0 & c & a-\lambda & b & 0 & \ldots \\ 0 & 0 & c & a-\lambda & b & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & c & a-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}.$$

First line:

$$(a-\lambda)x_1 + bx_2 = 0$$

$$x_2 = -\frac{(a-\lambda)}{b}x_1$$

Second line:

$$cx_1 + (a-\lambda)x_2 + bx_3 = 0$$

Upon substitution for x_2 we have

$$x_3 = \frac{1}{b}\left(c - \frac{(a-\lambda)^2}{b}\right)x_1,$$

continuing in this manner we obtain all coordinates. Our objective is to review the classical result on the eigenvalues. Several related examples will be worked in the sequence.

First, let us consider the general *Jacobi matrix* , all coefficients are real.

$$J_n = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \ldots & 0 \\ c_1 & a_2 & b_2 & 0 & \ldots & \ldots \\ 0 & c_2 & a_3 & b_3 & 0 & \ldots \\ 0 & 0 & c_3 & a_4 & b_4 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & c_{n-1} & a_n \end{bmatrix}.$$

Write $|J_n|$ for the determinant of the Jacobi matrix of $n-th$ order and evaluate it by the last row (or column):

$$|J_1| = |a_1|$$

$$|J_2| = \begin{vmatrix} a_1 & b_1 \\ c_1 & a_2 \end{vmatrix}$$

$$|J_3| = \begin{vmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{vmatrix} = a_3|J_2| - b_2c_2|J_1|$$

$$|J_4| = \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 \\ 0 & c_2 & a_3 & b_3 \\ 0 & 0 & c_3 & a_4 \end{vmatrix} = a_4|J_3| - b_3c_3|J_2|.$$

and so on. Clearly, there is a recursion for $n \geq 3$:

$$|J_n| = a_n|J_{n-1}| - a_{n-1}b_{n-1}|J_{n-2}|.$$

Next, we demonstrate how to transform the general Jacobi matrix into a symmetric matrix by a pair of diagonal matrices:

$$D_n = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & \dots & \dots \\ 0 & 0 & \alpha_3 & 0 & 0 & \dots \\ 0 & 0 & 0 & \alpha_4 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \alpha_{n-1} & 0 \\ 0 & 0 & 0 & \dots & \dots & \alpha_n \end{bmatrix}.$$

$$D_n^{-1} = \begin{bmatrix} \alpha_1^{-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2^{-1} & 0 & \dots & \dots & \dots \\ 0 & 0 & \alpha_3^{-1} & 0 & 0 & \dots \\ 0 & 0 & 0 & \alpha_4^{-1} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & \alpha_n^{-1} \end{bmatrix}.$$

α-s are yet undetermined.

$$DJ = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & \dots & \dots \\ 0 & 0 & \alpha_3 & 0 & 0 & \dots \\ 0 & 0 & 0 & \alpha_4 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & \alpha_n \end{bmatrix} \begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & 0 & \dots & \dots \\ 0 & c_2 & a_3 & b_3 & 0 & \dots \\ 0 & 0 & c_3 & a_4 & b_4 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_{n-1} & a_n \end{bmatrix} =$$

$$\begin{bmatrix} a_1\alpha_1 & b_1\alpha_1 & 0 & 0 & \dots & 0 \\ c_1\alpha_2 & a_2\alpha_2 & b_2\alpha_2 & 0 & \dots & \dots \\ 0 & c_2\alpha_3 & a_3\alpha_3 & b_3\alpha_3 & 0 & \dots \\ 0 & 0 & c_3\alpha_4 & a_4\alpha_4 & b_4\alpha_4 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_{n-1}\alpha_n & a_n\alpha_n \end{bmatrix};$$

Multiplying on the right by D^{-1}

$$\begin{bmatrix} a_1\alpha_1 & b_1\alpha_1 & 0 & 0 & \dots & 0 \\ c_1\alpha_2 & a_2\alpha_2 & b_2\alpha_2 & 0 & \dots & \dots \\ 0 & c_2\alpha_3 & a_3\alpha_3 & b_3\alpha_3 & 0 & \dots \\ 0 & 0 & c_3\alpha_4 & a_4\alpha_4 & b_4\alpha_4 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_{n-1}\alpha_n & a_n\alpha_n \end{bmatrix} \begin{bmatrix} \alpha_1^{-1} & 0 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2^{-1} & 0 & \dots & \dots & \dots \\ 0 & 0 & \alpha_3^{-1} & 0 & 0 & \dots \\ 0 & 0 & 0 & \alpha_4^{-1} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & \alpha_n^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} a_1\alpha_1\alpha_1^{-1} & b_1\alpha_1\alpha_2^{-1} & 0 & 0 & \dots & 0 \\ c_1\alpha_2\alpha_1 & a_2\alpha_2\alpha_2^{-1} & b_2\alpha_2\alpha_3^{-1} & 0 & \dots & \dots \\ 0 & c_2\alpha_3\alpha_2^{-1} & a_3\alpha_3\alpha_3^{-1} & b_3\alpha_3\alpha_4^{-1} & 0 & \dots \\ 0 & 0 & c_3\alpha_4\alpha_3^{-1} & a_4\alpha_4\alpha_4^{-1} & b_4\alpha_4\alpha_5^{-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c_{n-1}\alpha_{n-1}\alpha_n^{-1} & a_n\alpha_n\alpha_n^{-1} \end{bmatrix}.$$

The above matrix is symmetric if $x_{i,i+1} = x_{i+1,i}$, $i = 1, n-1$, where $x_{i,j}$ is an element. This condition gives us $n-1$ equations to solve. Without loss of generality we can set $\alpha_1 = 1$. The other equations will be solved by recursion:

$$b_1\alpha_1\alpha_2^{-1} = c_1\alpha_2\alpha_1^{-1}$$

$$\frac{b_1}{c_1} = \alpha_2^2 \Rightarrow \alpha_2 = \sqrt{\frac{b_1}{c_1}}$$

$$b_2\alpha_2\alpha_3^{-1} = c_2\alpha_3\alpha_2^{-1}$$

$$\frac{b_2}{c_2}a_2^2 = \alpha_3^2 \;\Rightarrow \alpha_3 = \sqrt{\frac{b_1 b_2}{c_1 c_2}} \;\text{ etc.}$$

$$\vdots$$

$$\alpha_n = \sqrt{\frac{b_1 b_2 \ldots b_{n-1}}{c_1 c_2 \ldots c_{n-1}}}.$$

Thus the concrete form of D is

$$D = \text{Diag}\left[1, \sqrt{\frac{b_1}{c_1}}, \sqrt{\frac{b_1 b_2}{c_1 c_2}} \ldots \sqrt{\frac{b_1 b_2 \ldots b_{n-1}}{c_1 c_2 \ldots c_{n-1}}}\right].$$

Further, it is easy to see that characteristic polynomials are equal

$$|D||D^{-1}| = 1$$

$$|D(J - \lambda I)D^{-1}| = |D||(J - \lambda I)||D^{-1}| = |(J - \lambda I)||D||D^{-1}| = |J - \lambda I|$$

$$|D(J - \lambda I)D^{-1}| = |DJD^{-1} - \lambda DID^{-1}| = |DJD^{-1} - \lambda I|$$

$$|J - \lambda I| = |DJD^{-1} - \lambda I|.$$

Therefore it is sufficient to calculate the eigenvalues for symmetrc DJD^{-1}. Now the common tridiagonal matrix is a special case of the general Jacobi matrix, so its eigenvalues can be determined by the appropiate symmetric tridiagonal matrix.

$$F = \text{Diag}\left[1, \sqrt{\frac{b}{c}}, \sqrt{\frac{bb}{cc}} \ldots \sqrt{\frac{bb \ldots b}{cc \ldots c}}\right].$$

$$FAF^{-1} = \begin{bmatrix} a & \sqrt{bc} & 0 & 0 & \ldots & 0 \\ \sqrt{bc} & a & \sqrt{bc} & 0 & \ldots & \ldots \\ 0 & \sqrt{bc} & a & \sqrt{bc} & 0 & \ldots \\ 0 & 0 & \sqrt{bc} & a & \sqrt{bc} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \sqrt{bc} & a \end{bmatrix}$$

Moreover, we know from the elements of matrix theory that symmetric matrices have real eigenvalues, and their eigenvectors associated with different eigenvalues are distinct.

End of Part I. To be continued.

20.3 Assignment 45.

- Combinatorial Analysis
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision April 22, 2020

Problems

I 25. The number of non-negative solutions of the Diophantine equation

$$x + 2y + 3z = n$$

is equal to the integer closest to

$$\frac{(n+3)^2}{12}.$$

Solution: The total number of non-negative solutions is equal to the coefficients of the n^{th} term of the product (convolution) :

$$(1 + x + x^2 + x^3 + ... + x^k + ...) \times (1 + x^2 + x^4 + ... + x^{2k} + ...)$$

$$\times (1 + x^3 + x^6 + ... + x^{3k} + ...) = \frac{1}{(1-x)(1-x^2)(1-x^3)}$$

$$= A_0 + A_1 x + ... + A_n x^n + ...$$

We present several methods for the calculation of A_n.

Claim I: The coefficients can be found by listing all partitions. We demonstrate it for small n, say up to $n = 10$, using lexiographic ordering, which in this case means 3's before 2's and 2's before 1's.

$$
\begin{aligned}
1 &= 1 \\
2 &= 2 = 1+1 \\
3 &= 3 = 2+1 = 1+1+1 \\
4 &= 3+1 = 2+2 = 2+1+1 = 1+1+1+1 \\
5 &= 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1 \\
6 &= 3+3 = 3+2+1 = 3+1+1+1 = 2+2+2 = 2+2+1+1 \\
&= 2+1+1+1+1 = 1+1+1+1+1+1 \\
7 &= 3+3+1 = 3+2+2 = 3+2+1+1 = 3+1+1+1+1 = 2+2+2+1 \\
&= 2+2+1+1+1 = 2+1+1+1+1+1 = 1+1+1+1+1+1+1 \\
8 &= 3+3+2 = 3+3+1+1 = 3+2+2+1 = 3+2+1+1+1 \\
&= 3+1+1+1+1+1 = 2+2+2+2 = 2+2+2+1+1 \\
&= 2+2+1+1+1+1 = 2+1+1+1+1+1+1 = 1+1+1+1+1+1+1+1 \\
9 &= 3+3+3 = 3+3+2+1 = 3+3+1+1+1 = 3+2+2+2 \\
&= 3+2+2+1+1 = 3+2+1+1+1+1 = 3+1+1+1+1+1+1 \\
&= 2+2+2+2+1 = 2+2+2+1+1+1 = 2+2+1+1+1+1+1 \\
&= 2+1+1+1+1+1+1+1 = 1+1+1+1+1+1+1+1+1 \\
10 &= 3+3+3+1 = 3+3+2+2 = 3+3+2+1+1 = 3+3+1+1+1+1 \\
&= 3+2+2+2+1 = 3+2+2+1+1+1 = 3+2+1+1+1+1+1 \\
&= 3+1+1+1+1+1+1+1+1+1+1 = 2+2+2+2+2 = 2+2+2+2+1+1 \\
&= 2+2+2+1+1+1+1 = 2+2+1+1+1+1+1+1 = 2+1+1+1+1+1+1+1+ \\
&= 1+1+1+1+1+1+1+1+1+1
\end{aligned}
$$

Summary:

n	A_n	n	A_n
1	1	6	7
2	2	7	8
3	3	8	10
4	4	9	12
5	5	10	14

Clearly, this method is not feasible for large n.

Claim II: The coefficients can be be calculated by recursion. Let $G(n;1,2)$ denote the generating function of the partitions of positive integer n into parts 1 and 2

$$G(n;1,2) = \frac{1}{(1-x)(1-x^2)} =$$

$$(1+x+x^2+x^3+...+x^k+...)\times(1+x^2+x^4+...+x^{2k}+...) =$$

$$1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+5x^8+5x^9+6x^{10}\dots$$

Then

$$(1-x^3)G(n;1,2,3) = G(n;1,2)$$

$$G(n;1,2,3) = G(n;1,2)\times(1+x^3+x^6+...+x^{3k}+...) =$$

$$(1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+\dots)\times(1+x^3+x^6+...+x^{3k}+...) =$$

$$1+x+2x^2+3x^3+4x^4+5x^5+7x^6+8x^7+9x^{12}+10x^{14}+\dots$$

The coefficients are the same as in **Claim I.**

Claim III: Calculation by matrix formalism, as discussed in an earlier Report (Q4,2014). Consider the generating function

$$A(x) = a_0 + a_1x + a_2x^2 + +a_3x^3 + \dots$$

and assign to it a certain lower triangular matrix of infinite dimensions. Moreover, let generating functions $B(x)$, $C(x)$ be assigned lower triangular matrices **B** and **C**, then the convolution of $A(x)*B(x)*C(x)$ can be obtained by matrix multiplication of **ABC**. In this concrete case the 11×11 segments are

$$A(x) = \frac{1}{(1-x)}$$

$$\mathbf{A_{11}} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0 \\ 1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0 \\ 1&1&1&1&1&1&1&0&0&0&0 \\ 1&1&1&1&1&1&1&1&0&0&0 \\ 1&1&1&1&1&1&1&1&1&0&0 \\ 1&1&1&1&1&1&1&1&1&1&0 \\ 1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix};$$

$$\mathbf{B_{11}} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&0&0&0 \\ 0&1&0&1&0&1&0&0&0&0&0 \\ 1&0&1&0&1&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0&0&0 \\ 1&0&1&0&1&0&1&0&1&0&0 \\ 0&1&0&1&0&1&0&1&0&1&0 \\ 1&0&1&0&1&0&1&0&1&0&1 \end{bmatrix};$$

$$\mathbf{C_{11}} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0&0\\ 1&0&0&1&0&0&1&0&0&0&0\\ 0&1&0&0&1&0&0&1&0&0&0\\ 0&0&1&0&0&1&0&0&1&0&0\\ 1&0&0&1&0&0&1&0&0&1&0\\ 0&1&0&0&1&0&0&1&0&0&1 \end{bmatrix};$$

$$\mathbf{D_{11}} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0&0\\ 2&1&1&0&0&0&0&0&0&0&0\\ 3&2&1&1&0&0&0&0&0&0&0\\ 4&3&2&1&1&0&0&0&0&0&0\\ 5&4&3&2&1&1&0&0&0&0&0\\ 7&5&4&3&2&1&1&0&0&0&0\\ 8&7&5&4&3&2&1&1&0&0&0\\ 10&8&7&5&4&3&2&1&1&0&0\\ 12&10&8&7&5&4&3&2&1&1&0\\ 14&12&10&8&7&5&4&3&2&1&1 \end{bmatrix};$$

The sought coefficients are the first column of D_{11} .

$$\begin{bmatrix} A_0\\ A_1\\ A_2\\ A_3\\ A_4\\ A_5\\ A_6\\ A_7\\ A_8\\ A_9\\ A_{10} \end{bmatrix} = \begin{bmatrix} 1\\ 1\\ 2\\ 3\\ 4\\ 5\\ 7\\ 8\\ 10\\ 12\\ 14 \end{bmatrix}.$$

Claim IV: Coefficient A_n can be determined by a simple computer program based on the algorithm for multiplication of a polynomial by another polynomial.

```
program xmpl_01
! Aufgaben und LehrSatze
! problem 25
! p  = series expansion of (1-x)**(-1)
! q  = series expansion of (1-x**2)**(-1)
! r  = series expansion of (1-x**3)**(-1)
! s  = work
! ut =terminal
implicit none
integer:: i,j,k,l,n
parameter (n=100)
real(kind=kind(1.0d0)):: z
real(kind=kind(1.0d0)), dimension (n+1) :: p(0:n), q(0:n), r(0:n), s(0:n)
!
do i=0,n
! set to 0
p(i)=0.0d0
q(i)=0.0d0
r(i)=0.0d0
s(i)=0.0d0
! initialize
p(i)=1.0d0
if (mod(i,2).eq.0) q(i)=1.0d0
if (mod(i,3).eq.0) r(i)=1.0d0
end do
!
do i=0,n
print '(5f7.2)' , float(i), p(i),q(i),r(i),s(i)
end do
!
write (*,*) 'init'
!
do i=0,n
do j=0,n
```

```
do k=0,n
l=i+j+k
if(l.ge.0.and.l.le.n) then
s(l)=s(l)+p(i)*q(j)*r(k)
endif
end do
end do
end do
!
do i=0,n,10
z=((float(i)+3.0)**2)/12.0
print '(3x,i5,f7.2,3x,f7.2)', i,s(i),z
end do


end program
```

Claim V: Analytical solution:

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1}{6(1-x)^3} + \frac{1}{4(1-x)^2} +$$

$$\frac{17}{72(1-x)} + \frac{1}{8(1+x)} + \frac{2+x}{9(1+x+x^2)}.$$

Proof of Claim V: Partial fraction decomposition by Bezout's method. Write

$$\left[(1-x)(1-x^2)(1-x^3)\right]^{-1} = \frac{A}{(1-x)^3} + \frac{B}{(1-x)^2} +$$

$$\frac{C}{(1-x)} + \frac{D}{(1+x)} + \frac{E+Fx}{(1+x+x^2)}.$$

Find A, B, C, D, E, F.

$$\frac{A}{(1-x)^3} = \frac{A(1+x)(1+x+x^2)}{(1-x)^3(1+x)(1+x+x^2)} = \frac{A(1+2x+2x^2+x^3)}{(1-x)^3(1+x)(1+x+x^2)}$$

$$\frac{B}{(1-x)^2} = \frac{B(1-x)(1+x)(1+x+x^2)}{(1-x)^2(1-x)(1+x)(1+x+x^2)} = \frac{B(1+x-x^3-x^4)}{(1-x)^2(1-x)(1+x)(1+x+x^2)}$$

$$\frac{C}{(1-x)} = \frac{C(1-x)^2(1+x)(1+x+x^2)}{((1-x)^3(1+x)(1+x+x^2)} = \frac{C(1-x^2-x^3+x^5)}{((1-x)^3(1+x)(1+x+x^2)}$$

$$\frac{D}{(1+x)} = \frac{D(1-x)^3(1+x+x^2)}{(1-x)^3(1+x)(1+x+x^2)} = \frac{D(1-2x+x^2-x^3+2x^4-x^5)}{(1-x)^3(1+x)(1+x+x^2)}$$

$$\frac{E+Fx}{(1+x+x^2)} = \frac{(E+Fx)(1-x)^3(1+x)}{(1-x)^3(1+x)(1+x+x^2)} = \frac{(E+Fx)(1-2x+2x^3-x^4)}{(1-x)^3(1+x)(1+x+x^2)}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & -2 & -2 & 1 \\ 2 & 0 & -1 & 1 & 0 & -2 \\ 1 & -1 & -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \frac{1}{6}; \quad B = \frac{1}{4}; \quad C = \frac{17}{72}; \quad D = \frac{1}{8}; \quad E = \frac{2}{9}; \quad F = \frac{1}{9}.$$

Further decomposition: Solution of quadratic equation $1+x+x^2$

$$w_{1,2} = \frac{-1 \pm \sqrt{1^2-4}}{2} = \frac{-1 \pm \imath\sqrt{3}}{2}; \quad w := e^{\frac{2\pi\imath}{3}} \quad w^2 := e^{\frac{4\pi\imath}{3}}$$

$$1+x+x^2 = (1-wx)(1-w^2x)$$

$$\frac{2+x}{9(1+x+x^2)} = \frac{2-(w+w^2)x}{9(1-wx)(1-w^2x)} = \frac{(1-w^2x)+(1-wx)}{9(x-w)(x-w^2)}$$

$$\frac{2+x}{9(1+x+x^2)} = \frac{1}{9(1-wx)} + \frac{1}{9(1-w^2x)}$$

Power series expansions:

$$\frac{1}{1-x} = \sum_{k=0} x^k; \quad \frac{1}{(1-x)^2} = \sum_{k=1} kx^{k-1}; \quad \frac{1}{(1-x)^3} = \sum_{k=2} \frac{k(k-1)}{2}x^{k-2}.$$

$$\frac{1}{1+x} = \sum_{k=0}(-1)^k x^k;$$

$$\frac{1}{9(1-wx)} = \frac{1}{9}(1+wx+w^2x^2+w^3x^3+\dots)$$

$$\frac{1}{9(1-w^2x)} = \frac{1}{9}(1 + w^2x + w^4x^2 + w^6x^3 + \dots)$$

The coefficients of x^k in the expansions of

$$\frac{1}{6(1-x)^3}; \quad \frac{1}{4(1-x)^2}; \quad \frac{17}{72(1-x)}$$

are

$$\frac{(k+2)(k+1)}{12}; \quad \frac{(k+1)}{4}; \quad \frac{17}{72};$$

respectively. The principal part is

$$\frac{(k+2)(k+1)}{12} + \frac{3(k+1)}{12} + \frac{24}{72} - \frac{7}{72} = \frac{(k^2+2k+k+2+3k+3+4)-7}{12} =$$

$$= \frac{(k^2+6k+9)-7}{12} = \frac{(k+3)^3}{12} - \frac{7}{72} < \frac{(k+3)^3}{12}.$$

The minor part is

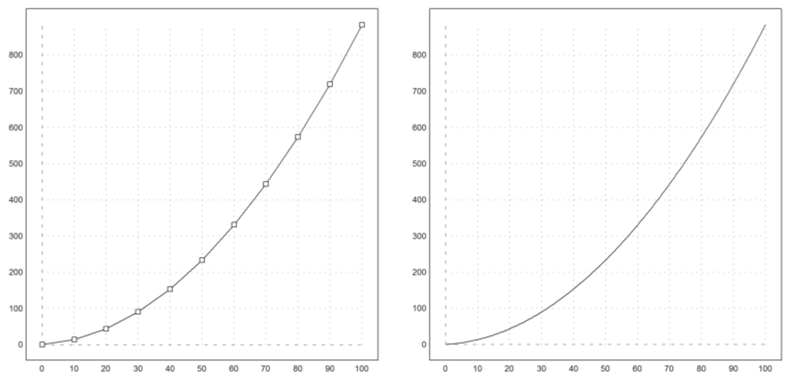
$$\left| -\frac{7}{12} + \frac{(-1)^k}{8} + \frac{2}{9}\cos\frac{2k\pi}{3} \right| \le \frac{32}{72} < \frac{1}{2}.$$

Therefore $A_n \approx \frac{(n+3)^3}{12}$ within $\frac{1}{2}$.

Solution:

n	$Sol'n$	$Approx.$
10	14	14.08
20	44	44.08
30	91	90.75
40	154	154.08
50	234	234.08
60	331	331.75
70	444	444.08
80	574	574.08
90	721	720.08
100	884	884.08

Graph of the calculated solution practically coincides with approximate analytical solution ;



End of demonstration. $\surd$