Pinter Consulting
Rough Copy
Draft, New Series No. 7.

J K Pinter, Dr.Tech.

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Motto

- Meg(g)y? Nem meg(g)y?
- Meg(g)y, de néha eröltetni kell az igényes matematikai továbbképzést.

Előszó

Meghatottan forgatta a vasrudat a parázsban. Valamit kérdezni akartam, de intett, hogy be kell tennie a szájpecket. Utána, óvatosan, a hátamon próbálta ki a vörösen izzó, szikrázó vasdarabot.

- Csúnya szaga van, ha pörkölődik - nevetett rám a szeme szögletéből. - Van, aki nem birja.

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Introduction

Pinter Consulting of Calgary, Alberta practices Mathematics, promotes clear thinking and offers Consultations, Tutorials and Seminars in Mathematics.

Contents

Chapter 7

Proceedings

7.1 Summary of Current Report

Private study for professional development:

Records of activities at Pinter Consulting : no extracurricular activities.

Collection of problems with our own solutions: Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis, with contributions from Choi, Lay, Riordan and Hua.

Socratic Programme

- Analysis
- Algebra and Number Theory
- Geometry
- Differential and Integral Equations

Continuos improvement, corrections and last revision March 9, 2015.

7.2 Assignment 22.

Summary

- Determinants and Quadratic Forms
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision March 9, 2015

Problems

VII 1 Form and calculate the determinants of the *adjacement matrices* of regular tetrahedron, hexahedron and octahedron.

Adjacement Matrices: Let the n vertices of the polyhedron be numbered in a definite order. Define the adjacement matrix $[a_{\lambda,\mu}]$ as follows: If vertices v_λ and v_μ are two end-points of an edge on the polyhedron then let $a_{\lambda,\mu} = a_{\mu,\lambda} = 1$. If vertices v_λ and v_μ are not two end-points of an edge then set $a_{\lambda,\mu} = a_{\mu,\lambda} = 0$. In particular, $a_{\lambda,\lambda} = 0$.

Regular polyhedra: A regular polyhedron is a polyhedron all of whose faces are regular polygons of the same shape and size. There are only five kinds of regular polyhedra: *tetrahedron, hexahedron (or cube), octahedron, icosa . . . , dodeca . . . .* Here, we are concerned with tetra-, hexa- , octahedron.

Figure	Face	Vertices	Edges	Faces	Faces around a Vertex
Tetra . . .	Equilat. triangle	4	6	4	3
Hexa . . .	Square	8	12	6	3
Octa . . .	Equilat. triangle	6	12	8	4

Tetrahedron:

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -\dfrac{3}{2} \end{vmatrix} = \left(-\frac{3}{2}\right) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -3.$$

Hexahedron:

$$\Delta_8 = \begin{vmatrix} 0&1&0&1&1&0&0&0\\ 1&0&1&0&0&1&0&0\\ 0&1&0&1&0&0&1&0\\ 1&0&1&0&0&0&0&1\\ 1&0&0&0&0&1&0&1\\ 0&1&0&0&1&0&1&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&1&1&0&1&0 \end{vmatrix} = \begin{vmatrix} 0&1&0&1&1&0&0&0\\ 1&0&1&0&0&1&0&0\\ 0&1&0&1&0&0&1&0\\ 0&0&0&0&0&-1&0&1\\ 0&0&-1&0&0&0&0&1\\ 0&1&0&0&1&0&1&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&1&1&0&1&0 \end{vmatrix} =$$

Second row subtracted from fourth and fifth rows;

$$= -\begin{vmatrix} 1&0&1&0&0&1&0&0\\ 0&1&0&1&1&0&0&0\\ 0&1&0&1&0&0&1&0\\ 0&0&0&0&0&-1&0&1\\ 0&0&-1&0&0&0&0&1\\ 0&1&0&0&1&0&1&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&1&1&0&1&0 \end{vmatrix} = -\begin{vmatrix} 1&0&1&0&0&1&0&0\\ 0&1&0&1&1&0&0&0\\ 0&0&0&0&-1&0&1&0\\ 0&0&0&0&0&-1&0&1\\ 0&0&-1&0&0&0&0&1\\ 0&0&0&-1&0&0&1&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&1&1&0&1&0 \end{vmatrix}$$

Inversion of first and second rows.First column clear. Second row subtracted from third and sixth. Second column clear.

$$-\begin{vmatrix} 1&0&1&0&0&1&0&0\\ 0&1&0&1&1&0&0&0\\ 0&0&0&0&-1&0&1&0\\ 0&0&0&0&0&-1&0&1\\ 0&0&0&0&0&1&0&2\\ 0&0&0&-1&0&0&1&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&1&1&0&1&0 \end{vmatrix} = \begin{vmatrix} 1&0&1&0&0&1&0&0\\ 0&1&0&1&1&0&0&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&0&0&-1&0&1\\ 0&0&0&0&0&1&0&2\\ 0&0&0&-1&0&0&1&0\\ 0&0&0&0&-1&0&1&0\\ 0&0&0&1&1&0&1&0 \end{vmatrix} =$$

Seventh row added to fifth.Inversion of seventh and third.

$$\begin{vmatrix} 1&0&1&0&0&1&0&0\\ 0&1&0&1&1&0&0&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&0&0&-1&0&1\\ 0&0&0&0&0&1&0&2\\ 0&0&0&0&1&0&2&0\\ 0&0&0&0&-1&0&1&0\\ 0&0&0&1&1&0&1&0 \end{vmatrix} = -\begin{vmatrix} 1&0&1&0&0&1&0&0\\ 0&1&0&1&1&0&0&0\\ 0&0&1&0&0&1&0&1\\ 0&0&0&1&1&0&1&0\\ 0&0&0&0&0&1&0&2\\ 0&0&0&0&1&0&2&0\\ 0&0&0&0&-1&0&1&0\\ 0&0&0&0&0&-1&0&1 \end{vmatrix} =$$

Eighth row added to sixth, inversion of eighth and fourth.

$$-\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 9.\surd$$

Octahedron:

$$\Delta_6 = \begin{vmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{vmatrix} = 0.\surd$$

The opposite vertices are labeled $(1,4),(2,5),(3,6)$.

VII 2

$$\det A = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ b_1 & a_1 & a_1 & \dots & a_1 \\ b_1 & b_2 & a_2 & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ b_1 & b_2 & b_3 & \dots & a_n \end{vmatrix} = (a_1-b_1)(a_2-b_2)\dots(a_n-b_n)$$

Proof: Case $n=2$:

$$\det A = \begin{vmatrix} 1 & 1 \\ b_1 & a_1 \end{vmatrix} = (a_1-b_1).$$

Case $n=3$:

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ b_1 & a_1 & a_1 \\ b_1 & b_2 & a_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ b_1-a_1 & a_1-a_1 & a_1-a_1 \\ b_1 & b_2 & a_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ b_1-a_1 & 0 & 0 \\ b_1-a_2 & b_2-a_2 & a_2-a_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ b_1-a_1 & 0 & 0 \\ b_1-a_2 & b_2-a_2 0 & \end{vmatrix}$$

Case $n = 4$:

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1 & a_1 & a_1 & a_1 \\ b_1 & b_2 & a_2 & a_2 \\ b_1 & b_2 & b_3 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1 - a_1 & a_1 - a_1 & a_1 - a_1 & a_1 - a_1 \\ b_1 & b_2 & a_2 & a_2 \\ b_1 & b_2 & b_3 & a_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1 - a_1 & 0 & 0 & 0 \\ b_1 - a_2 & b_2 - a_2 & a_2 - a_2 & a_2 - a_2 \\ b_1 & b_2 & b_3 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1 - a_1 & 0 & 0 & 0 \\ b_1 - a_2 & b_2 - a_2 & 0 & 0 \\ b_1 - a_3 & b_2 - a_3 & b_3 - a_3 & a_3 - a_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1 - a_1 & 0 & 0 & 0 \\ b_1 - a_2 & b_2 - a_2 & 0 & 0 \\ b_1 - a_3 & b_2 - a_3 & b_3 - a_3 & 0 \end{vmatrix} = (-1) \begin{vmatrix} b_1 - a_1 & 0 & 0 \\ b_1 - a_2 & b_2 - a_2 & 0 \\ b_1 - a_3 & b_2 - a_2 & b_3 - a_3 \end{vmatrix}$$

$$= -(b_1 - a_1)(b_2 - a_2)(b_3 - a_3) = (a_1 - b_1)(a_2 - b_2)(a_3 - b_3).$$

This shows that the theorem is true for $n = 4, 3, 2$. The general case follows.

$$\det A = \begin{vmatrix} 1 & 1 & \dots & \dots & 1 \\ b_1 & a_1 & \dots & & a_1 \\ b_1 & b_2 & a_2 & \dots & a_2 \\ b_1 & b_2 & b_3 & a_3 & \dots \\ b_1 & b_2 & \dots & b_{n-1} & a_{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1 - a_1 & a_1 - a_1 & a_1 - a_1 & a_1 - a_1 \\ b_1 & b_2 & a_2 & a_2 \\ b_1 & b_2 & b_3 & a_3 \end{vmatrix}$$

VII 3

$$\left| \frac{1}{a_\lambda + b_\mu} \right|_1^n = \frac{\Pi_{j>k}^{1,2,\dots n}(a_j - a_k)(b_j - b_k)}{\Pi_{\lambda,\mu}^{1,2,\dots n}(a_\lambda + b_\mu)}$$

Case $n = 2$:

$$\left| \frac{1}{a_\lambda + b_\mu} \right|_1^2 = \frac{(a_2 - a_1)(b_2 - b_1)}{(a_2 + b_1)(a_2 + b_2)(a_1 + b_1)(a_1 + b_2)}$$

$$\begin{vmatrix} \frac{1}{(a_1 + b_1)} & \frac{1}{(a_1 + b_2)} \\ \frac{1}{(a_2 + b_1)} & \frac{1}{(a_2 + b_2)} \end{vmatrix} = \begin{vmatrix} \frac{1}{(a_1 + b_1)} - \frac{1}{(a_2 + b_1)} & \frac{1}{(a_1 + b_2)} - \frac{1}{(a_2 + b_2)} \\ \frac{1}{(a_2 + b_1)} & \frac{1}{(a_2 + b_2)} \end{vmatrix} =$$

$$\begin{vmatrix} \frac{(a_2+b_1)-(a_1+b_1)}{(a_2+b_1)(a_1+b_1)} & \frac{(a_1+b_2)-(a_2+b_2)}{(a_2+b_2)(a_1+b_2)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} \end{vmatrix} =$$

$$\begin{vmatrix} \frac{a_2-a_1}{(a_2+b_1)(a_1+b_1)} & \frac{a_2-a_1}{(a_1+b_2)(a_2+b_2)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} \end{vmatrix} = \frac{a_2-a_1}{(a_2+b_1)(a_2+b_2)} \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} \\ 1 & 1 \end{vmatrix}$$

$$= \frac{a_2-a_1}{(a_2+b_1)(a_2+b_2)}\left(\frac{1}{(a_1+b_1)} - \frac{1}{(a_1+b_2)}\right) = \frac{a_2-a_1}{(a_2+b_1)(a_2+b_2)}\left(\frac{b_2-b_1}{(a_1+b_1)(a_1+b_2)}\right)$$

$$= \frac{(a_2-a_1)(b_2-b_1)}{(a_1+b_1)(a_1+b_2)(a_2+b_1)(a_2+b_2)} = \Delta_2 . \surd$$

Case $n = 3$:

$$\Delta_3 = \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & \frac{1}{(a_1+b_3)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & \frac{1}{(a_2+b_3)} \\ \frac{1}{(a_3+b_1)} & \frac{1}{(a_3+b_2)} & \frac{1}{(a_3+b_3)} \end{vmatrix} =$$

Subtract the $n-th$ row from the first $n-1$ rows.

$$\begin{vmatrix} \frac{1}{(a_1+b_1)} - \frac{1}{(a_3+b_1)} & \frac{1}{(a_1+b_2)} - \frac{1}{(a_3+b_2)} & \frac{1}{(a_1+b_3)} - \frac{1}{(a_3+b_3)} \\ \frac{1}{(a_2+b_1)} - \frac{1}{(a_3+b_1)} & \frac{1}{(a_2+b_2)} - \frac{1}{(a_3+b_2)} & \frac{1}{(a_2+b_3)} - \frac{1}{(a_3+b_3)} \\ \frac{1}{(a_3+b_1)} & \frac{1}{(a_3+b_2)} & \frac{1}{(a_3+b_3)} \end{vmatrix} =$$

Then by

$$\frac{1}{(a_k+b_i)} - \frac{1}{(a_3+b_i)} = \frac{a_3-a_k}{(a_k+b_i)(a_3+b_i)}; k = 1, 2;\ \ i = 1, 2, 3$$

$$\begin{vmatrix} \frac{(a_3-a_1)}{(a_1+b_1)(a_3+b_1)} & \frac{(a_3-a_1)}{(a_1+b_2)(a_3+b_2)} & \frac{(a_3-a_1)}{(a_1+b_3)(a_3+b_3)} \\ \frac{(a_3-a_2)}{(a_2+b_1)(a_3+b_1)} & \frac{(a_3-a_2)}{(a_2+b_2)(a_3+b_2)} & \frac{(a_3-a_2)}{(a_2+b_3)(a_3+b_3)} \\ \frac{1}{(a_3+b_1)} & \frac{1}{(a_3+b_2)} & \frac{1}{(a_3+b_3)} \end{vmatrix} =$$

$$\frac{(a_3-a_1)(a_3-a_2)}{(a_3+b_1)(a_3+b_2)(a_3+b_3)}\begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & \frac{1}{(a_1+b_3)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & \frac{1}{(a_2+b_3)} \\ 1 & 1 & 1 \end{vmatrix} = \Delta_3.$$

Next, subtract the $n-th$ column from the first $n-1$ columns.

$$\Delta_3 = [\frac{(a_3-a_1)(a_3-a_2)}{(a_3+b_1)(a_3+b_2)(a_3+b_3)}\begin{vmatrix} \frac{1}{(a_1+b_1)}-\frac{1}{(a_1+b_3)} & \frac{1}{(a_1+b_2)}-\frac{1}{(a_1+b_3)} & \frac{1}{(a_1+b_3)} \\ \frac{1}{(a_2+b_1)}-\frac{1}{(a_2+b_3)} & \frac{1}{(a_2+b_2)}-\frac{1}{(a_2+b_3)} & \frac{1}{(a_2+b_3)} \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_3 = [\frac{(a_3-a_1)(a_3-a_2)}{(a_3+b_1)(a_3+b_2)(a_3+b_3)}\begin{vmatrix} \frac{b_3-b_1}{(a_1+b_1)(a_1+b_3)} & \frac{b_3-b_2}{(a_1+b_2)(a_1+b_3)} & \frac{1}{(a_1+b_3)} \\ \frac{b_3-b_1}{(a_2+b_1)(a_2+b_3)} & \frac{b_3-b_2}{(a_2+b_2)(a_2+b_3)} & \frac{1}{(a_2+b_3)} \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_3 = [\frac{(a_3-a_1)(a_3-a_2)(b_3-b_1)(b_3-b_2)}{(a_1+b_3)(a_3+b_1)(a_3+b_2)(a_3+b_3)(a_2+b_3)}\begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & 1 \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_3 = \frac{(a_3-a_1)(a_3-a_2)(b_3-b_1)(b_3-b_2)}{(a_1+b_3)(a_3+b_1)(a_3+b_2)(a_3+b_3)(a_2+b_3)}\Delta_2$$

Reduction to $n=2$.

$$\Delta_3 = \frac{(a_3-a_1)(a_3-a_2)(b_3-b_1)(b_3-b_2)}{(a_1+b_3)(a_3+b_1)(a_3+b_2)(a_3+b_3)(a_2+b_3)}\frac{(a_2-a_1)(b_2-b_1)}{(a_1+b_1)(a_1+b_2)(a_2+b_1)(a_2+b_2)}\surd$$

The general case follows.The general case is as follows. Consider

$$\Delta_n = \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & \cdots & \frac{1}{(a_1+b_n)} \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & \cdots & \frac{1}{(a_2+b_n)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(a_n+b_1)} & \frac{1}{(a_n+b_2)} & \cdots & \frac{1}{(a_n+b_n)} \end{vmatrix}.$$

Subtract the last row from the preceeding rows and take out the following factors from the columns

$$\frac{1}{(a_n+b_1)},\ \frac{1}{(a_n+b_2)},\ \dots \frac{1}{(a_n+b_{n-1})},\ \frac{1}{(a_n+b_n)}$$

and the factors

$$(a_n-a_1),(a_n-a_2),\dots(a_n-a_{n-1}),1,$$

from the rows inthe same manner as in case $n=3$. Write

$$C_1^n = \frac{1}{(a_n+b_1)} \times \frac{1}{(a_n+b_2)} \times \dots \times \frac{1}{(a_n+b_{n-1})} \times \frac{1}{(a_n+b_n)}$$

and

$$C_2^n = (a_n-a_1)\times(a_n-a_2\times\dots\times(a_n-a_{n-1})\times 1.$$

Mutatis mutandis for the columns: in the remaining determinant subtract the last column from the preceeding columns and factor out

$$(b_n-b_1),(b_n-b_2),\dots(b_n-b_{n-1}),1$$

and

$$\frac{1}{(a_1+b_n)},\ \frac{1}{(a_2+b_n)},\ \dots \frac{1}{(a_{n-1}+b_n)},\ \frac{1}{(a_n+b_n)},$$

respectively. Write

$$C_3^n = (b_n-b_1)\times(b_n-b_2)\times\dots\times(b_n-b_{n-1})$$

and

$$C_4^n = \frac{1}{(a_1+b_n)} \times \frac{1}{(a_2+b_n)} \times \dots \frac{1}{(a_{n-1}+b_n)} \times \frac{1}{(a_n+b_n)}.$$

There remains a $(n-1)$-rowed corner minor of the given determinant

$$\Delta_n = C_1^n C_2^n C_3^n C_4^n \begin{vmatrix} \frac{1}{(a_1+b_1)} & \frac{1}{(a_1+b_2)} & \dots & \frac{1}{(a_1+b_{n-1})} & 1 \\ \frac{1}{(a_2+b_1)} & \frac{1}{(a_2+b_2)} & \dots & \frac{1}{(a_2+b_{n-1})} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(a_{n-1}+b_1)} & \frac{1}{(a_{n-1}+b_2)} & \dots & \frac{1}{(a_{n-1}+b_{n-1})} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Mathematical induction completes the proof.

Completion Write out the determinant for N.

7.3 Assignment 23.

Summary

- Combinatorial Analysis
- *Pólya - Szegő; Riordan*
- Last revision March 9, 2015

Problems

(1) Worked examples of the Vandermonde Theorem.

i) $(1+z)^n (1+z) = (1+z)^{(n+1)}$

$$(1+z)^n (1+z) = \sum_{k=0}^{n} \binom{n}{k} (z^k + z^{k+1}) = \sum_{k=0}^{n+1} \binom{n+1}{k} z^k.$$

Fix $k = m$, and compare the coefficients of z^m in the expansions above:

$$\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m}. \checkmark$$

ii) $(1+z)^n (1+z)^2 = (1+z)^{(n+2)}$

$$\left(\sum_{k=0}^{n} \binom{n}{k} z^k\right) \left(1 + 2z + z^2\right) = \sum_{k=0}^{n+2} \binom{n+2}{k} z^k$$

Compare coefficients of z^k

$$\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = \binom{n+2}{k}$$

or

$$\binom{2}{0}\binom{n}{k} + \binom{2}{1}\binom{n}{k-1} + \binom{2}{2}\binom{n}{k-2} = \binom{n+2}{k}. \checkmark$$

iii) General case $s \leq n$

$$(1+z)^n (1+z)^s = (1+z)^{(n+s)}$$

$$\left(\sum_{k=0}^{n}\binom{n}{k}z^k\right)\left(\sum_{k=0}^{s}\binom{s}{k}z^k\right)=\sum_{k=0}^{n+s}\binom{n+s}{k}z^k$$

Fix m,

$$0 < m \leq s;\ k+j=m$$

$$\sum_{k=0}^{m}\binom{n}{k}z^k\binom{s}{j}z^j=\sum_{k=0}^{m}\binom{n}{k}\binom{s}{j}z^{k+j}=z^m\left(\sum_{k=0}^{m}\binom{n}{k}\binom{s}{j}\right).$$

Again, comparing coefficients in the above expansions gives

$$\sum_{k=0}^{m}\binom{n}{k}\binom{s}{m-k}=\binom{n+s}{m}.$$

(2) Prove the following identities:

$$i)\ \ n\binom{n}{r}=(r+1)\binom{n}{r+1}+r\binom{n}{r}$$

$$ii)\binom{n}{2}\binom{n}{r}=\binom{r+2}{2}\binom{n}{r+2}+2\binom{r+1}{2}\binom{n}{r+1}+\binom{r}{2}\binom{n}{r}$$

$$iii)\binom{n}{s}\binom{n}{r}=\sum_{k=0}^{s}\binom{s}{k}\binom{r+s-k}{r-k}\binom{n}{r+s-k};\quad s\leq r$$

Solutions: The three questions may suggest an *inductive* argument since i) and ii) are special cases of iii). However we find that these cases can be proved by independent arguments as well.

$$i)\ \ n\binom{n}{r}=(r+1)\binom{n}{r+1}+r\binom{n}{r}$$

$$(n-r)\binom{n}{r}=(r+1)\binom{n}{r+1}$$

$$(n-r)\,\frac{n!}{r!(n-r)!}=(r+1)\,\frac{n!}{(r+1)!(n-r-1)!}$$

$$\frac{n!}{r!(n-r-1)!} = \frac{n!}{r!(n-r-1)!}\checkmark$$

ii) In anticipation of the general case iii) we re-write the right hand side of the equation

$$\binom{2}{0}\binom{r+2}{2}\binom{n}{r+2} + \binom{2}{1}\binom{r+1}{2}\binom{n}{r+1} + \binom{2}{2}\binom{r}{2}\binom{n}{r}$$

$$\binom{2}{0}\binom{r+2}{2}\binom{n}{r+2} = \frac{2!}{0!2!} \times \frac{(r+2)!}{2!r!} \times \frac{n!}{(n-2)!(n-r-2)!} =$$

$$\frac{2!}{0!2!} \times \frac{n(n-1)}{2!} \times \frac{(n-2)!}{r!(n-2-r)!} = \binom{2}{0}\binom{n}{2}\binom{n-2}{r}.$$

Applying the same method,

$$\binom{2}{1}\binom{r+2}{2}\binom{n}{r+2} = \binom{2}{1}\binom{n}{2}\binom{n-2}{r-1}.$$

$$\binom{2}{2}\binom{r}{2}\binom{n}{r} = \binom{2}{2}\binom{n}{2}\binom{n}{r-2}.$$

Summation,

$$\binom{2}{0}\binom{n}{2}\binom{n-2}{r} + \binom{2}{1}\binom{n}{2}\binom{n-2}{r-1} + \binom{2}{2}\binom{n}{2}\binom{n}{r-2} =$$

$$\binom{n}{2}\left[\binom{2}{0}\binom{n-2}{r} + \binom{2}{1}\binom{n-2}{r-1} + \binom{2}{2}\binom{n}{r-2}\right] = \binom{n}{2}\binom{n}{r}.$$

ii) Somewhat different proof: we can look at the last line as

$$\binom{n-2}{r} + 2\binom{n-2}{r-1} + \binom{n}{r-2} =$$

$$\binom{n-2}{r} + \binom{n-2}{r-1} + \binom{n-2}{r-1} + \binom{n-2}{r} =$$

$$\binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}.$$

iii) Having proved $s = 1, 2$ let us see the general case. Right hand side term T_k for k fixed is

$$T_k = \frac{s!}{k!(s-k)!} \times \frac{(r+s-k)!}{(r-k)!s!} \times \frac{n!}{(r+s-k)!(n-r-s+k)!}$$

$$= \frac{s!}{s!} \times \frac{1}{k!(s-k)!} \times \frac{1}{(r-k)!} \times \frac{n!}{(n-r-s+k)!}.$$

$$n! = (n-1)(n-2)\dots(n-s+1)(n-s)!$$

$$(n-r-s+k)! = ((n-s)-(r-k))!$$

$$T_k = \frac{s!}{s!} \times \frac{(n-1)(n-2)\dots(n-s+1)}{k!(s-k)!} \times \frac{(n-s)!}{(r-k)!((n-s)-(r-k))!} =$$

$$\frac{s!}{k!(s-k)!} \times \frac{(n-1)(n-2)\dots(n-s+1)}{s!} \times \frac{(n-s)!}{(r-k)!((n-s)-(r-k))!}.$$

Therefore

$$T_k = \begin{pmatrix} s \\ k \end{pmatrix} \begin{pmatrix} n \\ s \end{pmatrix} \begin{pmatrix} n-s \\ r-k \end{pmatrix},$$

$$\sum_{k=0}^{s} T_k = \begin{pmatrix} n \\ s \end{pmatrix} \sum_{k=0}^{s} \begin{pmatrix} s \\ k \end{pmatrix} \begin{pmatrix} n-s \\ r-k \end{pmatrix} = \begin{pmatrix} n \\ s \end{pmatrix} \begin{pmatrix} n \\ r \end{pmatrix}.$$

where

$$\sum_{k=0}^{s} \begin{pmatrix} s \\ k \end{pmatrix} \begin{pmatrix} n-s \\ r-k \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix}.$$

The last identity is also known as

$$\sum_{k=0}^{m} \begin{pmatrix} x \\ k \end{pmatrix} \begin{pmatrix} y \\ m-k \end{pmatrix} = \begin{pmatrix} x+y \\ m \end{pmatrix}.$$

Prove the other 3 identities from Riordan.

I 31.1 Prove that

$$\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n-1 \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$$

Solution This statement was already proved in the preamble. A different proof follows. Mark one object (*) out of n objects. When selecting r objects out of n objects the marked object(*) is either selected or not. If selected then there are $(r-1)$ more to be chosen out of $(n-1)$, or if it is not selected then there are all r to be chosen out of $(n-1)$.

I 31.2

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots + (-1)^r \binom{n}{r} = (-1)^r \binom{n-1}{r}.$$

Solution We demonstrate the solution by an example, $n = 6$.

$$\binom{6}{0} = \binom{5}{0} \checkmark$$

$$\binom{6}{0} - \binom{6}{1} = (-1)\binom{5}{1} \checkmark$$

$$\binom{6}{0} - \binom{6}{1} + \binom{6}{2} = (-1)^2 \binom{5}{2}; 1 - 6 + 15 = 10\checkmark$$

$$\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} = \binom{5}{2} - \binom{6}{3}$$

$$\binom{5}{2} - \binom{6}{3} = -\binom{5}{3}$$

$$\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} = -\binom{5}{3} \checkmark$$

Let $n \geq 2$. The identity is true for $r = 0, 1, 2$ $r < n - 1$:

$$\binom{n}{0} = \binom{n-1}{0}$$

$$\binom{n}{0} - \binom{n}{1} = -\binom{n-1}{1}$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} = \binom{n-1}{2}.$$

Suppose we have

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots + (-1)^r \binom{n}{r} = (-1)^r \binom{n-1}{r}.$$

Adding the same term on both sides

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots + (-1)^r \binom{n}{r} + (-1)^{r+1} \binom{n}{r+1} =$$

$$(-1)^r \binom{n-1}{r} + (-1)^{r+1} \binom{n}{r+1} = (-1)^{r+1} \binom{n-1}{r+1}$$

by **(31.1)** . Thus we pass from r to $r+1$.

I 32

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Solution

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n},$$

by

$$\binom{n}{k}\binom{n}{n-k}.$$

$$(1+z)^n (1+z)^n = (1+z)^{2n}$$

7.4 Assignment 24.

- Combinatorial Analysis
- *Hua, Riordan*
- Last revision March 9, 2015

Notes on partition Let q be real or complex, $|q| < 1$ and let us define the following functions

$$\begin{aligned} q_0 &= \prod_{n=1}^{\infty}(1-q^{2n}) \\ q_1 &= \prod_{n=1}^{\infty}(1+q^{2n}) \\ q_2 &= \prod_{n=1}^{\infty}(1+q^{2n-1}) \\ q_3 &= \prod_{n=1}^{\infty}(1-q^{2n-1}). \end{aligned}$$

Recall that the infinite product

$$(1+u_1)(1+u_2)(1+u_3)\ldots = \prod_{k=1}^{\infty}(1+u_k)$$

converges to $P \neq 0$ if $\lim P_n = P$ where

$$P_n = (1+u_1)(1+u_2)(1+u_3)\ldots(1+u_n), \quad u_k \neq -1, \forall k.$$

A necessary and sufficient condition condition that $\prod(1+u_k)$ converge absolutely is that $\sum u_k$ converge absolutely.

Proposition 1. If $|q| < 1$, then

$$q_1 q_2 q_3 = 1.$$

Proof (1): Consider index sets $\{n\}, \{2n-1\}, \{2n\}$:

$$\{2n\} \cup \{2n-1\} = \{n\}, \ \{2n\} \cap \{2n-1\} = \{\emptyset\}$$

$$q_0 q_3 = \prod_{n=1}^{\infty}(1-q^{2n}) \prod_{n=1}^{\infty}(1-q^{2n-1}) = \prod_{n=1}^{\infty}(1-q^n).$$

$$q_1 q_2 = \prod_{n=1}^{\infty}(1+q^{2n}) \prod_{n=1}^{\infty}(1+q^{2n-1}) = \prod_{n=1}^{\infty}(1+q^n)$$

$$q_0 q_1 q_2 q_3 = \prod_{n=1}^{\infty}(1-q^n) \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}(1-q^{2n}) = q_0$$

$$q_1 q_2 q_3 = 1.$$

Proof (2):

$$\begin{aligned} q_2 q_3 &= \left(\prod_{n=1}^{\infty}(1+q^{2n-1})\right)\left(\prod_{n=1}^{\infty}(1-q^{2n-1})\right) \\ &= \prod_{n=1}^{\infty}\Big((1+q^{2n-1})(1-q^{2n-1})\Big) \\ &= \prod_{n=1}^{\infty}(1-q^{2(2n-1)}) \end{aligned}$$

$$\begin{aligned} q_1 &= \prod_{n=1}^{\infty}(1+q^{2n}) \\ &= \prod_{n=1}^{\infty}(1+q^{2(2n-1)}) \prod_{n=1}^{\infty}(1+q^{4(2n-1)}) \prod_{n=1}^{\infty}(1+q^{8(2n-1)}) \dots \end{aligned}$$

because

$$\{2n\} = \{2(2n-1)\} \cup \{4(2n-1)\} \cup \{8(2n-1)\} \dots$$

To verify this decomposition let M be an even number, $M \epsilon \{2n\}$. Then

$$M = 2^{\alpha} p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$$

where $\alpha > 0$ and $p_i, i = 1, \dots k$ are odd primes. This representation is unique. Thus M belongs to one and only one subset, $\{2^\alpha(2n-1)\}$ and $p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k} = 2m - 1$ for some integer m. On the other hand, there is no odd number in any of the $\{2^\alpha(2n-1)\}$ subsets. Therefore the decomposition is valid. Here is a numerical example of "taking out the powers of 2" from even numbers up to 40:

$$\{2n\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \dots 40, \dots\}$$

$$\{2(2n-1)\} = \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, \dots\}$$

$$\{4(2n-1)\} = \{4, 12, 20, 28, 36, 44, \dots\}$$

$$\{8(2n-1)\} = \{8, 24, 40, 48, \dots\}$$

$$\{16(2n-1)\} = \{16, 48, \dots\}$$

$$\{32(2n-1)\} = \{32, 96, \dots\}$$

$$\{64(2n-1)\} = \{64, \dots\}$$

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \dots 40\} =$$

$$\{2, 6, 10, 14, 18, 22, 26, 30, 34, 38\} \cup \{4, 12, 20, 28, 36\} \cup \{8, 24, 40\} \cup \{16\} \cup \{32\}.$$

End of numerical example.

$$\begin{aligned}
q_1 q_2 q_3 &= \prod_{n=1}^{\infty}(1 - q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \dots \\
&= \left(\prod_{n=1}^{\infty}(1 - q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{2(2n-1)})\right) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)}) \dots \\
&= \left(\prod_{n=1}^{\infty}(1 - q^{4(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)})\right) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \dots \\
&= \left(\prod_{n=1}^{\infty}(1 - q^{8(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)})\right) \prod_{n=1}^{\infty}(1 + q^{16(2n-1)}) \dots = 1.
\end{aligned}$$

The lowest exponent of q is rising yet the infinite products have the same limit.Why? Write

$$K(q) = \prod_{n=1}^{\infty}(1 - q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{2(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \dots$$

$$K(q^2) = \prod_{n=1}^{\infty}(1 - q^{4(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{4(2n-1)} \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \dots$$

$$K(q^4) = \prod_{n=1}^{\infty}(1 - q^{8(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{8(2n-1)}) \prod_{n=1}^{\infty}(1 + q^{16(2n-1)}) \dots$$

$$K(q) = K(q^2) = K(q^4) = \dots = K(0) = 1$$

$K(\ )$ is invariant under the substitution of q^2 for q, $\lim q^{2n} = 0$.

7.5 Assignment 25.

Summary

- Determinants and Quadratic Forms
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision March 9, 2015

Problems

VII 8 Set $\Delta = ad - cd$. Then the functional determinant

$$\frac{\partial(a\Delta, b\Delta, c\Delta, d\Delta)}{\partial(a,b,c,d)} = 3\Delta^4.$$

(Jacobian determinant, differentiability assumed throughout.)

Lemma(Muir):

$$\Lambda = \frac{\partial\left(\phi f_1, \phi f_2, \ldots, \phi f_n\right)}{\partial\left(x_1, x_2, \ldots, x_n\right)} = \phi^{n-1} \begin{vmatrix} \phi & f_1 & f_2 & \ldots & f_n \\ \frac{-\partial\phi}{\partial x_1} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \ldots & \frac{\partial f_n}{\partial x_1} \\ \frac{-\partial\phi}{\partial x_2} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_n}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{-\partial\phi}{\partial x_n} & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \ldots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

Proof:

$$\Lambda = \phi^{-1} \begin{vmatrix} \phi & f_1 & f_2 & \ldots & f_n \\ -\phi\frac{\partial\phi}{\partial x_1} & \phi\frac{\partial f_1}{\partial x_1} & \phi\frac{\partial f_2}{\partial x_1} & \ldots & \phi\frac{\partial f_n}{\partial x_1} \\ -\phi\frac{\partial\phi}{\partial x_2} & \phi\frac{\partial f_1}{\partial x_2} & \phi\frac{\partial f_2}{\partial x_2} & \ldots & \phi\frac{\partial f_n}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\phi\frac{\partial\phi}{\partial x_n} & \phi\frac{\partial f_1}{\partial x_n} & \phi\frac{\partial f_2}{\partial x_n} & \ldots & \phi\frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

Next, multiply first row by $\dfrac{\partial\phi}{\partial x_i}$ to obtain

$$\left[\ \phi\frac{\partial\phi}{\partial x_i} \quad f_1\frac{\partial\phi}{\partial x_i} \quad f_2\frac{\partial\phi}{\partial x_i} \quad \dots \quad f_n\frac{\partial\phi}{\partial x_i}\ \right]$$

and add it two $i+1-st$ row for $i = 1, 2, \dots n$.

$$\Lambda = \phi^{-1}\begin{vmatrix} \phi & f_1 & f_2 & \dots & f_n \\ \phi\frac{\partial\phi}{\partial x_1} - \phi\frac{\partial\phi}{\partial x_1} & f_1\frac{\partial\phi}{\partial x_1} + \phi\frac{\partial f_1}{\partial x_1} & \phi\frac{\partial f_2}{\partial x_1} + f_2\frac{\partial\phi}{\partial x_1} & \dots & \phi\frac{\partial f_n}{\partial x_1} + f_n\frac{\partial\phi}{\partial x_1} \\ \phi\frac{\partial\phi}{\partial x_2} - \phi\frac{\partial\phi}{\partial x_2} & f_1\frac{\partial\phi}{\partial x_2} + \phi\frac{\partial f_1}{\partial x_2} & \phi\frac{\partial f_2}{\partial x_2} + f_2\frac{\partial\phi}{\partial x_2} & \dots & \phi\frac{\partial f_n}{\partial x_2} + f_n\frac{\partial\phi}{\partial x_2} \\ \dots & \dots & \dots & \dots & \dots \\ \phi\frac{\partial\phi}{\partial x_n} - \phi\frac{\partial\phi}{\partial x_n} & f_1\frac{\partial\phi}{\partial x_n} + \phi\frac{\partial f_1}{\partial x_n} & \phi\frac{\partial f_2}{\partial x_n} + f_2\frac{\partial\phi}{\partial x_n} & \dots & \phi\frac{\partial f_n}{\partial x_n} + f_n\frac{\partial\phi}{\partial x_n} \end{vmatrix}$$

$$\Lambda = \phi^{-1}\begin{vmatrix} \phi & f_1 & f_2 & \dots & f_n \\ 0 & f_1\frac{\partial\phi}{\partial x_1} + \phi\frac{\partial f_1}{\partial x_1} & \phi\frac{\partial f_2}{\partial x_1} + f_2\frac{\partial\phi}{\partial x_1} & \dots & \phi\frac{\partial f_n}{\partial x_1} + f_n\frac{\partial\phi}{\partial x_1} \\ 0 & f_1\frac{\partial\phi}{\partial x_2} + \phi\frac{\partial f_1}{\partial x_2} & \phi\frac{\partial f_2}{\partial x_2} + f_2\frac{\partial\phi}{\partial x_2} & \dots & \phi\frac{\partial f_n}{\partial x_2} + f_n\frac{\partial\phi}{\partial x_2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & f_1\frac{\partial\phi}{\partial x_n} + \phi\frac{\partial f_1}{\partial x_n} & \phi\frac{\partial f_2}{\partial x_n} + f_2\frac{\partial\phi}{\partial x_n} & \dots & \phi\frac{\partial f_n}{\partial x_n} + f_n\frac{\partial\phi}{\partial x_n} \end{vmatrix}$$

$$\Lambda = (\phi^{-1}\phi)\begin{vmatrix} f_1\frac{\partial\phi}{\partial x_1} + \phi\frac{\partial f_1}{\partial x_1} & \phi\frac{\partial f_2}{\partial x_1} + f_2\frac{\partial\phi}{\partial x_1} & \dots & \phi\frac{\partial f_n}{\partial x_1} + f_n\frac{\partial\phi}{\partial x_1} \\ f_1\frac{\partial\phi}{\partial x_2} + \phi\frac{\partial f_1}{\partial x_2} & \phi\frac{\partial f_2}{\partial x_2} + f_2\frac{\partial\phi}{\partial x_2} & \dots & \phi\frac{\partial f_n}{\partial x_2} + f_n\frac{\partial\phi}{\partial x_2} \\ \dots & \dots & \dots & \dots \\ f_1\frac{\partial\phi}{\partial x_n} + \phi\frac{\partial f_1}{\partial x_n} & \phi\frac{\partial f_2}{\partial x_n} + f_2\frac{\partial\phi}{\partial x_n} & \dots & \phi\frac{\partial f_n}{\partial x_n} + f_n\frac{\partial\phi}{\partial x_n} \end{vmatrix},$$

which,in fact, is the Jacobian. End of proof of Lemma.

Solution: Substitution

$$\phi = ad - bc;\ \ f_1 = a; f_2 = b; f_3 = c; f_= d.$$

$$\frac{\partial(a\Delta, b\Delta, c\Delta, d\Delta)}{\partial(a,b,c,d)} = \Delta^3\begin{vmatrix} ad-bc & a & b & c & d \\ -d & 1 & 0 & 0 & 0 \\ c & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ -a & 0 & 0 & 0 & 1 \end{vmatrix} = \Delta^3\left(\Delta + ad - cb - cb + ad\right) = 3\Delta^4.$$

VII 9 Vandermonde variations. Caluculate

$$i) V_n = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}; ii) \bar{V_n}^{(1)} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ x_1^3 & x_2^3 & \dots & x_n^3 \\ \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n \end{vmatrix};$$

$$iii) \bar{V_n}^{(n-1)} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ x_1^n & x_2^n & \dots & x_n^n \end{vmatrix}; iv) \bar{V_n}^{(s)} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{s-1} & x_2^{s-1} & \dots & x_n^{s-1} \\ x_1^{s+1} & x_2^{s+1} & \dots & x_n^{s+1} \\ \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n \end{vmatrix}.$$

The first is the standard Vandermonde determinant. The other three have a lacuna in the sequence of the powers of $x_i, i = 1, 2, \dots n$.

Solution of i) Induction on n. Note that for $n = 2$

$$\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1.$$

Upon multiplying the $i - th$ row by $-x_1$ and adding it to the $i + 1st$ row we have for $i = 1, 2, \dots n - 1$

$$V_n = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 - x_1 & x_2 - x_1 & \dots & x_n - x_1 \\ x_1^2 - x_1^2 & x_2^2 - x_1 x_2 & \dots & x_n^2 - x_1 x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} - x_1^{n-1} & x_2^{n-1} - x_1 x_2^{n-2} & \dots & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & x_2 - x_1 & \dots & x_n - x_1 \\ 0 & x_2^2 - x_1 x_2 & \dots & x_n^2 - x_1 x_n \\ \dots & \dots & \dots & \dots \\ 0 & x_2^{n-1} - x_1 x_2^{n-2} & \dots & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & \dots & x_n - x_1 \\ x_2^2 - x_1 x_2 & \dots & x_n^2 - x_1 x_n \\ \dots & \dots & \dots \\ x_2^{n-1} - x_1 x_2^{n-2} & \dots & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix} =$$

$$\begin{vmatrix} (x_2-x_1) & \dots & (x_n-x_1) \\ (x_2-x_1)x_2 & \dots & (x_n-x_1)x_n \\ \dots & \dots & \dots \\ (x_2-x_1)x_2^{n-2} & \dots & (x_n-x_1)x_n^{n-2} \end{vmatrix} = (x_2-x_1)\dots(x_n-x_1) \begin{vmatrix} 1 & \dots & 1 \\ x_2 & \dots & x_n \\ \dots & \dots & \dots \\ x_2^{n-2} & \dots & x_n^{n-2} \end{vmatrix}.$$

The last determinant has the *common factors* $(x_j - x_1), j = 2, 3, \dots n$ taken out from each column and it is of order $n-1$. Using the same method, the reduction yields that the Vandermonde determinant of order n is the *product of all possible differences* of $(x_i - x_k), n \geq i > k \geq 1$; $V_n = \Pi(x_i - x_k)$.

Solution of ii) We will calculate $\bar{V}_n^{(1)}$ by augmenting it with a extra row and an extra column. Consider

$$\bar{V}_n^{(1} \Rightarrow \begin{vmatrix} 1 & 1 & \dots & 1 & \downarrow \\ \rightarrow * & * & \dots & * & * \\ x_1^2 & x_2^2 & \dots & x_n^2 & * \\ x_1^3 & x_2^3 & \dots & x_n^3 & * \\ \dots & \dots & \dots & \dots & * \\ x_1^n & x_2^n & \dots & x_n^n & * \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & z \\ x_1^2 & x_2^2 & \dots & x_n^2 & z^2 \\ x_1^3 & x_2^3 & \dots & x_n^3 & z^3 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n & z^n \end{vmatrix} = D$$

Determinant D is a Vandermonde determinant

$$D = \Pi(x_i - x_k)\Pi(z - x_j);\; i = 1, 2 \dots n, i > k \geq 1; j = 1, 2 \dots n$$

vhich can be expanded as a polynomial in z. Write

$$A = \Pi(x_i - x_k)$$

$$\Pi(z - x_j) = z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \alpha_2 z^{n-2} \dots + \alpha_{n-1} z + \alpha_n$$

where by *Vieta's formulas* (a. k. a. *elementary symmetric polynomials*)

$$\alpha_1 = -(x_1 + x_2 + x_3 + \dots + x_n)$$

$$\alpha_2 = x_1x_2 + x_1x_3 + \dots + x_1x_n + x_2x_3 + \dots + x_{n-1}x_{n-2}$$

$$\dots$$

$$a_{n-1} = (-1)^{n-1}(x_1x_2x_3 \dots x_{n-1} + x_1x_2 \dots x_{n-2}x_{n-1} + \dots + x_2 \dots x_{n-1}x_n)$$

$$a_n = (-1)^n(x_1x_2x_3 \dots x_{n-1}x_n).$$

Thus

$$D(\mathbf{z}) = \mathrm{Az^n} + \mathrm{A}\alpha_1\mathrm{z^{n-1}} + \mathrm{A}\alpha_2\mathrm{z^{n-2}} + \mathrm{A}\alpha_2\mathrm{z^{n-2}} \ldots + \mathrm{A}\alpha_{\mathrm{n-1}}\mathrm{z} + \mathrm{A}\alpha_{\mathrm{n}}.$$

Further, $D(\mathbf{z})$ can be expanded by the last column:

$$D(\mathbf{z}) = \begin{vmatrix} 1 & 1 & \ldots & 1 & 1 \\ x_1 & x_2 & \ldots & x_n & z \\ x_1^2 & x_2^2 & \ldots & x_n^2 & z^2 \\ x_1^3 & x_2^3 & \ldots & x_n^3 & z^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^n & x_2^n & \ldots & x_n^n & z^n \end{vmatrix}_{\mathrm{(n+1)\times(n+1)}} = (-1)^{\mathrm{n+1}} \begin{vmatrix} x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ x_1^3 & x_2^3 & \ldots & x_n^3 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^n & x_2^n & \ldots & x_n^n \end{vmatrix}_{\mathrm{n\times n}}$$

$$+(-1)^{n+2}\mathbf{z} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ x_1^3 & x_2^3 & \ldots & x_n^3 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^n & x_2^n & \ldots & x_n^n \end{vmatrix}_{\mathrm{n\times n}} + (-1)^{\mathrm{n+3}}\mathbf{z^2} \begin{vmatrix} x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ x_1^3 & x_2^3 & \ldots & x_n^3 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^n & x_2^n & \ldots & x_n^n \end{vmatrix}_{\mathrm{n\times n}} \ldots$$

$$+(-1)^{n+1+s}\mathbf{z^s} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{s-1} & x_2^{s-1} & \ldots & x_n^{s-1} \\ x_1^{s+1} & x_2^{s+1} & \ldots & x_n^{s+1} \\ \ldots & \ldots & \ldots & \ldots \\ x_1^n & x_2^n & \ldots & x_n^n \end{vmatrix}_{\mathrm{n\times n}} \ldots$$

$$+(-1)^{2n}\mathbf{z^{n-1}} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ x_1^n & x_2^n & \ldots & x_n^n \end{vmatrix}_{\mathrm{n\times n}} + (-1)^{\mathrm{(2n+1)}}\mathbf{z^n} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ x_1^3 & x_2^3 & \ldots & x_n^3 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}_{\mathrm{n\times n}} .$$

The comparing of the coefficients of the two polynomials resolves the remaining qustions.

Example:

$$\hat{D}_3(z) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & z \\ x_1^2 & x_2^2 & x_3^2 & z^2 \\ x_1^3 & x_2^3 & x_n^3 & z^3 \end{vmatrix} = - \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_n^3 \end{vmatrix} + z \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_n^3 \end{vmatrix}$$

$$-z^2 \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^3 & x_2^3 & x_n^3 \end{vmatrix} + z^3 \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}.$$

$$\tilde{D}_3(z) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & z \\ x_1^2 & x_2^2 & x_3^2 & z^2 \\ x_1^3 & x_2^3 & x_n^3 & z^3 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)(z - x_1)(z - x_2)(z - x_3).$$

$$A = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

$$(z - x_1)(z - x_2)(z - x_3) = z^3 - z^2(x_3 + x_2 + x_1) + z(x_3x_2 + x_3x_1 + x_2x_1) - x_3x_2x_1$$

$$\tilde{D}_3(z) = Az^3 - A(x_3 + x_2 + x_1)z^2 + A(x_3x_2 + x_3x_1 + x_2x_1)z - Ax_3x_2x_1.$$

$$\hat{D}_3(z) = \tilde{D}_3(z)$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_n^3 \end{vmatrix} = x_3x_2x_1(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_n^3 \end{vmatrix} = (x_3x_2 + x_3x_1 + x_2x_1)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^3 & x_2^3 & x_n^3 \end{vmatrix} = (x_3 + x_2 + x_1)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

VII 9 Prove that $\rho - 2$ is divisor of the expression

$$D = \begin{vmatrix} \rho & \frac{l}{m}+\frac{m}{l} & \frac{n}{l}+\frac{l}{n} \\ \frac{l}{m}+\frac{m}{l} & \rho & \frac{m}{n}+\frac{n}{m} \\ \frac{n}{l}+\frac{l}{n} & \frac{m}{n}+\frac{n}{m} & \rho \end{vmatrix}; \quad l \neq 0, \quad m \neq 0, \quad n \neq 0.$$

Determine the other factors.

Lemma Curios identity for $l \neq 0, \quad m \neq 0, \quad n \neq 0.$

$$\left(\frac{m^2+n^2}{mn}\right)^2 + \left(\frac{l^2+m^2}{lm}\right)^2 + \left(\frac{n^2+l^2}{nl}\right)^2 - \left(\frac{l^2+m^2}{lm}\right)\left(\frac{m^2+n^2}{mn}\right)\left(\frac{n^2+l^2}{nl}\right) = 4.$$

Proof of Lemma

$$\left(\frac{m^2+n^2}{mn}\right)^2 + \left(\frac{l^2+m^2}{lm}\right)^2 + \left(\frac{n^2+l^2}{nl}\right)^2 = 6 + \left(\frac{m^2}{n^2}+\frac{n^2}{m^2}\right) + \left(\frac{l^2}{m^2}+\frac{m^2}{l^2}\right) + \left(\frac{n^2}{l^2}+\frac{l^2}{n^2}\right).$$

$$\left(\frac{l^2+m^2}{lm}\right)\left(\frac{m^2+n^2}{mn}\right)\left(\frac{n^2+l^2}{nl}\right) = 2 + \left(\frac{m^2}{n^2}+\frac{n^2}{m^2}\right) + \left(\frac{l^2}{m^2}+\frac{m^2}{l^2}\right) + \left(\frac{n^2}{l^2}+\frac{l^2}{n^2}\right).\checkmark$$

Solution We present two diferent proofs. If $D = D(\rho)$ is considered a polynomial in ρ, then by the well-known theorem on factorization of polynomials $D(2) = 0$ implies that $(\rho - 2)$ divides (or is a linear factor of) D.

First, we use facts from the *theory of quadratic forms* to show $D(2) = 0$. D is the determinant of the matrix associated with quadratic form Q:

$$Q(x,y,x) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \rho & \frac{l}{m}+\frac{m}{l} & \frac{n}{l}+\frac{l}{n} \\ \frac{l}{m}+\frac{m}{l} & \rho & \frac{m}{n}+\frac{n}{m} \\ \frac{n}{l}+\frac{l}{n} & \frac{m}{n}+\frac{n}{m} & \rho \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$\rho x^2 + \rho y^2 + \rho z^2 + 2\left(\frac{l}{m}+\frac{m}{l}\right)xy + 2\left(\frac{n}{l}+\frac{l}{n}\right)xz + 2\left(\frac{m}{n}+\frac{n}{m}\right)yz =$$

$$\rho(x^2+y^2+z^2) - 2(x^2+y^2+z^2) + 2\left(\frac{x}{l}+\frac{y}{m}+\frac{z}{n}+\right)(lx+my+nz).$$

$$Q=(\rho-2)\left(x^2+y^2+z^2\right)+2\left(\frac{x}{l}+\frac{y}{m}+\frac{z}{n}+\right)(lx+my+nz)\,.$$

Therefore, if $\rho=2$

$$Q=2\left(\frac{x}{l}+\frac{y}{m}+\frac{z}{n}+\right)(lx+my+nz)\,,$$

so Q is the *product of two linear forms* . This means that the rank of the matrix associated with Q is 2 or 1; so it is *singular* . In fact, we can classify Q:

$$Q=\frac{1}{2}\left[\left(\frac{1}{l}+l\right)x+\left(\frac{1}{m}+m\right)y+\left(\frac{1}{n}+n\right)z\right]^2$$

$$-\frac{1}{2}\left[\left(\frac{1}{l}-l\right)x+\left(\frac{1}{m}-m\right)y+\left(\frac{1}{n}-n\right)z\right]^2,$$

or Q is written as a sum of two squares, one with positive and one with negative coefficients. Therefore $D=0$ at $\rho=2$.

Next, we take a direct approach and calculate D as a polynomial in ρ:

$$D=\begin{vmatrix}\rho & \frac{l}{m}+\frac{m}{l} & \frac{n}{l}+\frac{l}{n}\\ \frac{l}{m}+\frac{m}{l} & \rho & \frac{m}{n}+\frac{n}{m}\\ \frac{n}{l}+\frac{l}{n} & \frac{m}{n}+\frac{n}{m} & \rho\end{vmatrix}=\begin{vmatrix}\rho & \frac{l^2+m^2}{lm} & \frac{n^2+l^2}{nl}\\ \frac{l^2+m^2}{lm} & \rho & \frac{m^2+n^2}{nm}\\ \frac{n^2+l^2}{nl} & \frac{m^2+n^2}{nm} & \rho\end{vmatrix}=$$

$$\rho\begin{vmatrix}\rho & \frac{m^2+n^2}{nm}\\ \frac{m^2+n^2}{nm} & \rho\end{vmatrix}-\frac{l^2+m^2}{lm}\begin{vmatrix}\frac{l^2+m^2}{lm} & \frac{m^2+n^2}{nm}\\ \frac{n^2+l^2}{nl} & \rho\end{vmatrix}$$

$$+\frac{n^2+l^2}{nl}\begin{vmatrix}\frac{l^2+m^2}{lm} & \rho\\ \frac{n^2+l^2}{nl} & \frac{m^2+n^2}{nm}\end{vmatrix}=\rho\left(\rho^2-\left[\frac{m^2+n^2}{nm}\right]^2\right)-$$

$$\frac{l^2+m^2}{lm}\left(\frac{l^2+m^2}{lm}\rho-\frac{m^2+n^2}{nm}\frac{n^2+l^2}{nl}\right)+\frac{n^2+l^2}{nl}\left(\frac{l^2+m^2}{lm}\frac{m^2+n^2}{nm}-\rho\frac{n^2+l^2}{nl}\right)=$$

$$\rho^3-\rho\left[\left(\frac{l^2+m^2}{lm}\right)^2+\left(\frac{m^2+n^2}{nm}\right)^2+\left(\frac{n^2+l^2}{nl}\right)^2\right]+2\left[\left(\frac{l^2+m^2}{lm}\right)\left(\frac{m^2+n^2}{nm}\right)\left(\frac{n^2+l^2}{nl}\right)\right].$$

$$D(2) = 2^3 - 2\left[\left(\frac{l^2+m^2}{lm}\right)^2 + \left(\frac{m^2+n^2}{nm}\right)^2 + \left(\frac{n^2+l^2}{nl}\right)^2\right] +$$

$$2\left[\left(\frac{l^2+m^2}{lm}\right)\left(\frac{m^2+n^2}{nm}\right)\left(\frac{n^2+l^2}{nl}\right)\right] =$$

$$8 - 2\left[\left(\frac{l^2+m^2}{lm}\right)^2 + \left(\frac{m^2+n^2}{nm}\right)^2 + \left(\frac{n^2+l^2}{nl}\right)^2 - \left(\frac{l^2+m^2}{lm}\right)\left(\frac{m^2+n^2}{nm}\right)\left(\frac{n^2+l^2}{nl}\right)\right] = 0,$$

by the Lemma.

Check proof, find a better one.

7.6 Assignment 26.

Summary

- Determinants and Quadratic Forms
- *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis,*
- Last revision March 9, 2015

Problems

Hilbert matrix

$$A_n = \left[\frac{1}{i+j-1}\right]_{i,j=1}^{i,j=n}$$

Write out the determinant for $n = 2, 3, 4, 5$. Write out the inverse for $n = 2, 3, 4, 5$. (Choi, AMM, Vol 90, No. 5, May 1983, pp 301-312)

This is a computer programming exercise. Verify Choi's formulas.

7.7 Assignment 27.

- Geometry
- *Lay: Convex Sets and their Applications*
- Last revision March 9, 2015

Definitions

The linear (vector) space R^n together with the *inner product* is the *n-dimensional Euclidean space* E^n.

$$x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$y = (\beta_1, \beta_2, \ldots, \beta_n)$$

$$x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$$

$$\lambda x = (\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n)$$

$$\langle x, y \rangle = \sum_{i=1}^{n} \alpha_i \beta_i$$

$\alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_n$ are real numbers, λ is a real number, a scalar; all elements of R; x, y are vectors, elements of E^n. Moreover, the origin (of the assumed coordinate system) is

$$\theta = (0, 0, 0, \ldots 0),$$

the empty set is denoted $\emptyset$. The norm of a vector, denoted by $\|x\|$, is defined by

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

Problems

1.1. *Properties of standard inner product:* x, y, z vectors in E^n, α real.

(*a*) $\langle x, x\rangle \geq 0;\ \langle x, x\rangle = 0\ iff\ x = \theta$

$$\langle x, x\rangle = \sum_{i=1}^{n} \alpha_i^2 \geq 0.$$

(*b*) $\langle x, y\rangle = \sum_{i=1}^{n} \alpha_i\beta_i = \sum_{i=1}^{n} \beta_i\alpha_i = \langle y, x\rangle.$

(*c*) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

$$\sum_{i=1}^{n}(\alpha_i + \beta_i)\gamma_i = \sum_{i=1}^{n}(\alpha_i\gamma_i + \beta_i\gamma_i) = \sum_{i=1}^{n} \alpha_i\gamma_i + \sum_{i=1}^{n} \beta_i\gamma_i$$

(*d*) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$

$$\sum_{i=1}^{n} \lambda\alpha_i\beta_i = \lambda\sum_{i=1}^{n} \alpha_i\beta_i$$

1.2. *Properties of norm:*

(*a*) $\|x\| > 0;\ x \neq \theta,\ \|\theta\| = 0.$

(*b*) $\|\alpha x\| = |\alpha|\|x\|.$

(*c*) $\|x + y\| \leq \|x\| + \|y\|$

(*d*) $\langle x, y\rangle = \|x\|\|y\| \cos\gamma$

Proof: Only the *Cauchy-Schwarz inequality* will be examined.

$$x = (\alpha_1, \alpha_2, \ldots \alpha_n);\ \ y = (\beta_1, \beta_2, \ldots \beta_n); \lambda \text{real}$$

$$(\alpha_i \lambda + \beta_i)^2 \geq 0 \ \forall i$$

$$\sum_{i=1}^{n} (\alpha_i \lambda + \beta_i)^2 \geq 0$$

This inequality is true for all λ .

$$A^2\lambda^2 + 2C\lambda + B^2 \geq 0,$$

$$A^2 = \sum_{i=1}^{n} \alpha_i^2;\ \ B^2 = \sum_{i=1}^{n} \beta_i^2;\ \ C = \sum_{i=1}^{n} \alpha_i \beta_i$$

$$\lambda^2 + \frac{2C}{A^2}\lambda + \frac{B^2}{A^2} \geq 0$$

$$\left(\lambda + \frac{C}{A^2}\right)^2 + \frac{B^2}{A^2} - \frac{C^2}{A^4} \geq 0$$

Choose

$$\lambda = -\frac{C}{A^2}.$$

Then

$$\left(\lambda + \frac{C}{A^2}\right)^2 = 0,$$

therefore

$$\frac{B^2}{A^2} - \frac{C^2}{A^4} \geq 0,$$

and

$$C^2 \leq A^2 B^2.$$

$$\left(\sum_{i=1}^{n} \alpha_i \beta_i\right)^2 = \left(\sum_{i=1}^{n} \alpha_i^2\right) \left(\sum_{i=1}^{n} \beta_i^2\right)$$

Taking the square roots of both sides gives

$$\left|\sum_{i=1}^{n}\alpha_i\beta_i\right| \leq \sqrt{\left(\sum_{i=1}^{n}\alpha_i^2\right)}\sqrt{\left(\sum_{i=1}^{n}\beta_i^2\right)},$$

or

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

The absolute value of the inner product of two vectors x, y is less than equal to the product of the norms (lengths) of the vectors.

$$\langle x, y\rangle = \|x\|\|y\|\cos\gamma,$$

where $\cos\gamma$ is the angle between vectors x, y. The angle so defined coincides with $\cos\gamma$ in two and three dimensional spaces.

Next, we review the triangle-inequality :

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

$$2\langle x, y\rangle \leq |2\langle x, y\rangle| \leq 2\|x\|\|y\|$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$\|x+y\|^2 = (\|x\| + \|y\|)^2$$

$$\|x+y\| = \|x\| + \|y\|.$$

1.3. *Prove the following for all real α and all sets A, B, C*

$$(a)\ (A+B)+C = A+(B+C)$$

$$A+B = \{x+y : x\epsilon A \wedge y\epsilon B\}$$

$$(A+B)+C = \{(x+y)+z : (x+y)\epsilon A+B \wedge z\epsilon C\}$$

$$A+B+C = \{x+y+z : x\epsilon A \wedge y\epsilon B \wedge z\epsilon C\}$$

$$A+B+C = (A+B)+C = A+(B+C)$$

vector addition is associative.

(*b*) $\alpha(A+B)=\alpha A+\alpha B$

$\alpha A=\{\alpha x : x\epsilon A\}; \quad \alpha B=\{\alpha y : y\epsilon B\}$

$\alpha(A+B)=\{\alpha(x+y) : (x+y)\epsilon A+B\}$

$\alpha(x+y)=\alpha x+\alpha y$

$\alpha(A+B)=\alpha A+\alpha B$

scalar multiplication is distributive over vector addition.

1.4. *In E^2, let A_1 be the closed line segment from the origin to $(0,2)$, A_2 be the closed line segment from the origin to $(2,0)$, A_3 be the closed line segment from the origin to $(2,2)$, respectively. Let $A_4=B(\theta,1)$, open ball with center θ radius 1. Describe*

(*a*) A_1+A_2

$A_1+A_2=\{x+y : x\epsilon A_1 \wedge y\epsilon A_2\}$

$x=\lambda(0,1);\ 0\le\lambda\le 2; y=\mu(1,0);\ 0\le\mu\le 2$

A_1+A_2 is a closed square with vertices $(0,0),(0,2),(2,2),(2,0)$.

(*b*) A_1+A_4

$A_1+A_4=\{B(\lambda(0,1),1), 0\le\lambda\le 2\}$

(*c*) $(A_1+A_2)+A_3$

translate of (A_1+A_2) by $(2,2)$.

1.5. *Prove that each of the following is an open set.*

(a) An open ball $B(Q, \rho)$.

Clearly, Q is an interior point because

$$B(Q, \frac{\rho}{2}) \subset B(Q, \rho).$$

Let $P \neq Q$ be a point in $B(Q, \rho)$. Then

$$0 < d(P, Q) < \rho$$

Set $\sigma = \rho - d(P, Q)$ and consider $B(p, \frac{s}{2})$.We claim

$$B(P, \frac{\sigma}{2}) \subset B(Q, \rho)$$

Let R be a point in $B(P, \frac{\sigma}{2})$ then

$$d(Q, R) \leq d(Q, P) + d(P, R) = d(Q, P) + \frac{\sigma}{2} < d(Q, P) + \sigma = \rho.$$

This proves that $R \epsilon B(Q, \rho)$ and P is an interior point of $B(Q, \rho)$. Therefore any point of $B(Q, \rho)$ is an interior point.

(b) E^n .

If $P \epsilon E^n$ then $B(P, \rho) \subset E^n$ for $\rho > 0$.

(c) $\emptyset$.

Suppose it is not open. Then there is a point P and a radius $\rho > 0$ such that $P \ \epsilon \ \emptyset$, and $B(P, \rho)$ is not in $\emptyset$. Contradiction: $\emptyset$ cannot contain P.

(d) The union of any collection of open sets.

Suppose

$$\Lambda = \cup A_\gamma, \ \gamma \epsilon \Gamma$$

where Γ is any index set and A_γ's are open sets. Let $P \epsilon A_\gamma$ for some γ. Since A_γ is an open set there is a $B(P, \rho)$ such that

$$B(P, \rho) \subset A_\gamma \subset \Lambda.$$

(e) The intersection of any finite collection of open sets. Write

$$\Lambda = \cap A_\gamma, \ \gamma \epsilon \Gamma$$

where Γ is a finite index set and A_γ's are open sets. If $P \ \epsilon \ \Lambda$ then $P \ \epsilon \ A_\gamma$ for $\forall \gamma \ \epsilon \ \Gamma$. For every A_γ there exists an open ball centered at P such that the open ball is contained in A_γ. Since there are only a finite number of open balls under consideration, we can find one with a minimal radius ρ_0. Set $\delta = \frac{\rho_0}{2}$. Then

$$B(P, \delta) \subset A_\gamma, \ \gamma \epsilon \Gamma.$$

Thus there exists an open ball centered at P which is contained in Λ, and P is an interior point of Λ.

1.6. *Prove that each of the following is a closed set.*

(a) Any finite set.

Consider $\Pi = \{P_i, \ i = 1, \dots n, P_i \epsilon E^n\}$ We want to show that the complement of Π

$$\tilde{\Pi} = \{x : x \ \epsilon \ E^n \wedge x \neq P_i\}$$

is open. Take $Q \epsilon \tilde{\Pi}$. Then there are finite number of positive distances between Q and P_i's. Let let us find their minimum.

$$\rho = \min\{d(Q, P_1), d(Q, P_2), \dots d(Q, P_n)\}.$$

Set $\sigma = B(Q, \frac{\rho}{2})$. Then

$$B(Q, \frac{\rho}{2}) \ \subset \tilde{\Pi}$$

which shows that $\tilde{\Pi}$ is an open set. Therefore Π is closed.

(b) E^n.

What is the complement of E^n ? It is $\emptyset$. We just showed in **1.5.c** that the $\emptyset$ is open. Therefore E^n is closed.

(c) $\emptyset$. The complement of $\emptyset$ is E^n, which is known to be an open set by **1.5.b** . Therefore $\emptyset$ is a closed set.

(d) The intersection of any collection of closed sets.

Suppose

$$\Lambda = \cap A_\gamma, \ \gamma \epsilon \Gamma$$

where Γ is any index set and A_γ's are closed sets. By De Morgan's identity

$$\tilde{\Lambda} = \cup \tilde{A}_\gamma, \ \gamma \epsilon \Gamma$$

where $\tilde{A}_\gamma, \ \gamma \epsilon \Gamma$ are open sets. The union of any collection of open sets is an open set by **1.5.d** . Therefore $\tilde{\Lambda}$ is an open set, and Λ is a closed set.

(e) The union of any finite collection of closed sets.
Write

$$\Lambda = \cup A_\gamma,\ \gamma \epsilon \Gamma$$

where Γ is a finite index set and A_γ's are closed sets. The method is the same as above:

$$\tilde{\Lambda} = \cap \tilde{A}_\gamma,\ \gamma \epsilon \Gamma$$

by de Morgan's identity, where $\tilde{A}_\gamma$'s are all open because the complement of a closed set is open. The intersection of any finite collection of open sets is an open set by **1.5.e** . Therefore $\tilde{\Lambda}$ is an open set, and Λ is a closed set.

1.7. Suppose $A \subset B$. Prove the following
(a) $int(A) \subset int(B)$.
The interior of A is the union of all the open sets contained in A. Let S be an open set in A.

$$S \subset A \subset B$$

by construction. Therefore every open set S of A is an open set of B. Consider any collection of open sets in A. Any collection of open sets in A is an open set in A and therefore in B. The union of all the open sets contained in A is an open set in A and therefore in B. Thus $int(A)$ is an open set in B and

$$int(A) \subset int(B)$$

since $int(B)$ is the union of all the open sets contained in B.
(b) $cl(A) \subset cl(B)$
The closure of A is the intersection of all the closed sets containing A. Let

$$\Lambda = \{U : U \subset E^n \wedge U \text{ closed set} \wedge A \subset U\}.$$

Then

$$cl(A) = \cap U,\ U \epsilon \Lambda.$$

But $cl(B)$ is a closed set that contains A because $B \subset cl(B)$, and A is contained in B. Therefore $cl(B)$ is a member of Λ.

1.8. Prove the following:

(a) The interior of a set A is the set of all interior points of A.

By definition, the interior of a set A, $int(A)$, is the union of all the open sets contained in A.

First, we show that an interior point is in the interior of A. Let P be an interior point of A. Then there exists an open ball $B(P, \rho)$ such that

$$B(P, \rho) \subset A$$

therefore

$$\begin{array}{rl} i) & P \epsilon B(P, \rho) \\ ii) & B(P, \rho) \subset A \\ \hline & B(P, \rho) \subset \ int(A) \\ & P \ \epsilon \ int(A). \end{array}$$

Next, let P be a point in the interior of A. We want to show that P is an interior point of A. If P is a point in the interior of A then there exists an open set G such that

$$P \epsilon G \subset \ int(A) \subset A.$$

Then P is an interior point of G and so it is an interior point of A.

Remark:

$$int(A) \subset A \subset cl(A)$$

The interior of A is the "largest" open set contained in A, the closure of A is the "smallest" closed set that contains A.

(b) A point x is in $cl(A)$ if and only if for every $\delta > 0$, the open ball $B(x, \delta)$ contains at least one point of A.

Discussion: A point x in E^n is called a *contact point* of set $A \subset E^n$ if every open ball $B(x, \rho)$ contains at least one point of A.

A point x in E^n is called a *limit point* of set $A \subset E^n$ if every open ball $B(x, \rho)$ contains infinitely many points of A. Point x may or may not belong to A.

A point x in E^n is called a *isolated point* of set $A \subset E^n$ if point x belongs to A and in an open ball $B(x, \rho)$ with sufficiently small radius contains no other point of A than x it self.

Obviously, an isolated point is not a limit point, and a contact point can be ether a limit point or an isolated point. A set A is closed if it contains all of its limit points. Furthermore, $cl(A)$ is the set of all contact points of A. Next, we prove that this definition coincides with the one previously given: the closure of A is the intersection of all closed sets containing A.

$$\begin{array}{rl} i) & P\epsilon cl(A) \\ ii) & \exists U, U \subset E^n, U\text{closed} \\ iii) & A \subset U \\ iv) & P\epsilon\tilde{U} \\ \hline & \exists\rho : B(P,\rho) \cap A = \emptyset \end{array}$$

Let, therefore, P be a contact point of A, U a typical closed subset of E^n that contains A, and suppose that P belongs to the complement of U. The complement of U is an open set because U is closed. If P is in the complement of U than there exists an open ball that belongs entirely to the complement of U, thus has no points in common with U. and a fortiori no points in common with A. But then P is not a contact point. A BIG, FAT, CONTRADICTION! Therefore the assumption that P is in the complement of U is false. P is in U, and a contact point of A is a point of a typical closed set U containing A.

To prove the converse, let P belong to every closed set U that contains A. Suppose P is not a contact point of A. Consider a typical closed set U that contains A. P is an element of A by construction. By assumption, P is not a contact point, thus there exist an open ball $B(P,\rho)$ that has no point of A. Consider the set V which is U with a smaller ball around P removed:

$$V = \{x : x\epsilon U \wedge x\neg\epsilon B(P, \frac{\rho}{2})\}.$$

Now take $cl(V)$, it is a closed set and it does not contain P, but it has A as a subset. Thus P does not belong to every closed set that contains A. Contradiction. Therefore P is a contact point of A.

7.8 Miscellaneous Notes

7.8.1 Current interests

Spartan Old School Tutorials: 3 levels, undergraduate math standards with basic programming, Latex typesetting.

Classics in Pure Math: *Pólya - Szegő: Aufgaben und Lehrsätze aus der Analysis, Konvexer Körper , Vinogradov-Turán, Riordan, Hua*

Classics in Applied Math : *Ciarlet, Birkhoff-Rota, Geÿza Freud*

Mathematical Modelling and Numerical Analysis: *Smith, Morton- Mayers, Ames*

7.8.2 Envoy

" Footprints? "
" Footprints. "
" A man's or a woman's? "
" Dr. Mortimer looked strangely at us for an instant, and his voice sank almost to a whisper as he answered:"
" Mr. Holmes, they were the footprints of a gigantic hound. "